Goal

In the context of differential equations:
Given set of solution $\subseteq$ full set of solutions
Does equality hold? “Measure” the size of the solution set.
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Given set of solution $\subseteq$ full set of solutions (in $\mathbb{C}[[x_1, \ldots, x_n]]$)
Does equality hold? “Meassure” the size of the solution set.
Example: Burgers’ equation

\[ \frac{\partial^2}{\partial x^2} u(t, x) = -\frac{\partial}{\partial t} u(t, x) - 2u(t, x)\frac{\partial}{\partial x} u(t, x) \]

Ansatz \( u(t, x) = \sum_{i,j} a_{i,j} \frac{t^i x^j}{i!j!} \)
Example: Burgers’ equation

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Ansatz \( u(t, x) = \sum_{i,j} a_{i,j} \frac{t^i x^j}{i! j!} \implies a_{0,2} = -a_{1,0} - 2a_{0,0}a_{0,1}. \)
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\( 2\ell + 1 \) free coefficients up to order \( \ell \).
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Theorem (Dimension polynomial, Kolchin)

Let \( I \) be a differential prime ideal.

1. There is a dimension polynomial \( \omega_I(\ell) \in \mathbb{Q}[\ell] \), which ultimately describes the number of free coefficients up to order \( \ell \).
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2. (Further properties)
The dimension polynomial

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2. (Further properties)

Let $J \supseteq I$ be another differential prime ideal.

3. $I = J$ iff $\omega_I = \omega_J$. 
(Radical) differential ideals represent differential equations.
Number of free coefficients \( \cong \) dimension of the residue class ring.
Prime decomposition of the differential ideal.
Decomposition into characterizable differential ideals suffices.

**Theorem (Dimension polynomial)**

Let \( I \) be a characterizable differential ideal.

1. There is a dimension polynomial \( \omega_I(\ell) \in \mathbb{Q}[\ell] \), which ultimatively describes the number of free coefficients up to order \( \ell \).

2. (Further properties)

Let \( J \supseteq I \) be another characterizable differential ideal.

3. \( I = J \) iff \( \omega_I = \omega_J \) and the degrees of the equations in the simple differential systems, which describe \( I \) and \( J \), are equal.
Example: Burgers’ equation (cont.)

\[
\frac{\partial^2}{\partial x^2} u(t, x) = -\frac{\partial}{\partial t} u(t, x) - 2u(t, x) \frac{\partial}{\partial x} u(t, x)
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The dimension polynomial is \(2\ell + 1\).
Example: Burgers’ equation (cont.)

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\]

The dimension polynomial is \(2\ell + 1\).

Maple:

\[
\left\{ c_1 - \tanh(c_1 x + c_2 t + c_3) - \frac{c_2}{2c_1} \left| c_1, c_2, c_3 \in \mathbb{C}, c_1 \neq 0 \right. \right\}
\]
Problems of the dimension polynomial

- Decomposition necessary; dimension polynomial not additive.
- Imprecise: only dimension.
- Prescribing initial values for differential equations impossible.
- Non-generic center of expansion:
  \[ t \cdot \frac{\partial}{\partial t} f(t) = 1 \text{ around } t = 0. \]
Algebraic counting polynomial $c$

Defined for constructible sets.

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Algebraic counting polynomial $c$

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2. $c(A^1) = \infty$ (formal symbol in the polynomial ring $\mathbb{Z}[\infty]$)
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1. \( c(V) = |V| \) if \( V \) finite
2. \( c(\mathbb{A}^1) = \infty \) (formal symbol in the polynomial ring \( \mathbb{Z}[\infty] \))
3. \( c(U \cup V) = c(U) + c(V) \)
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3. $c(U \sqcup V) = c(U) + c(V)$
4. $c(F) = c(B) \cdot c(\text{fiber})$ if well defined for coordinate proj. $F \rightarrow B$. 

Example: 

$c(V) = c(V) + c(V)$ by (3)
$c(V) = c(V) + 1$ by (1)
$c(V) = 2 \cdot c(V) + 1$ by (1), (4)
$c(V) = 2 \cdot \infty - 1$ by (1), (2), (3)

Well-defined, algorithmic, $\deg \infty(c(V)) = \dim(V)$.

Theorem (Plesken) Let $U \subseteq V$ constructible. Then $U = V$ iff $c(U) = c(V)$. 

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Example

$\begin{align*}
c(\begin{array}{c}
\end{array}) &= c(\begin{array}{c}
\end{array}) + c(\begin{array}{c}
\end{array}) \\
&= c(\begin{array}{c}
\end{array}) + 1 \\
&= 2 \cdot c(\begin{array}{c}
\end{array}) + 1 \\
&= 2 \cdot \infty - 1 \\
&= 2 \cdot \infty
\end{align*}$
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Example

\[
\begin{align*}
c(\text{——————}) &= c(\text{——————}) + c(\cdot) \quad \text{by (3)} \\
&= c(\text{——————}) + 1 \quad \text{by (1)}
\end{align*}
\]
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c\left(\begin{array}{c}
\end{array}\right) = c\left(\begin{array}{c}
\end{array}\right) + c\left(\begin{array}{c}
\end{array}\right) \quad \text{by (3)}
\]
\[
= c\left(\begin{array}{c}
\end{array}\right) + 1 \quad \text{by (1)}
\]
\[
= 2 \cdot c\left(\begin{array}{c}
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= 2 \cdot (\infty - 1) + 1 = 2 \cdot \infty - 1 \quad \text{by (1), (2), (3)}
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  \_ \\
  \_ \\
  \_ \\
  \_ \\
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  \_ \\
  \_ \\
  \_ \\
  \_ \\
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**Theorem (Plesken)**

Let $U \subseteq V$ constructible. Then $U = V$ iff $c(U) = c(V)$.
Differential counting polynomial

Idea: Counting polynomial for power series coefficients up to order $\ell$. 

\begin{align*}
\text{Theorem} & \quad \text{There exists a suitable countably infinite decomposition.}
\end{align*}

\begin{align*}
\text{Definition} & \quad \text{Applying the five axioms for the counting polynomial to the decomposition yields the } \ell\text{-th differential counting polynomial in } \mathbb{Z}[\infty, \aleph_0] \text{ (for every order } \ell \geq 0). \\
\text{Closed form:} & \quad \text{differential counting polynomial (if ultimately correct).}
\end{align*}

\begin{align*}
\text{Examples:} & \quad \infty^{2\ell+1} \infty^3 - \infty^{2\ell+1} + (\ell + 1) \infty^{\ell - 1} \\
\end{align*}
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**Theorem**

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There exists a suitable countably infinite decomposition.

$$c(\mathbb{A}^1 \setminus \text{countably infinite set}) = \infty - \mathbb{N}_0 \in \mathbb{Z}[\infty, \mathbb{N}_0]$$
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**Definition**

Applying the five axioms for the counting polynomial to the decomposition yields the $\ell$-th differential counting polynomial in $\mathbb{Z}[\infty, \aleph_0]$ (for every order $\ell \geq 0$).

Closed form: **differential counting polynomial** (if ultimatively correct).
Differential counting polynomial

Idea: Counting polynomial for power series coefficients up to order \( \ell \).

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**Definition**

Applying the five axioms for the counting polynomial to the decomposition yields the \( \ell \)-th **differential counting polynomial** in \( \mathbb{Z}[\infty, \mathbb{N}_0] \) (for every order \( \ell \geq 0 \)).

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**Examples:**

\[
\begin{align*}
\infty^{2\ell+1} \\
\infty^3 - \infty^2 + \infty - \mathbb{N}_0 \\
\infty^{\ell+2} - \infty^{\ell+1} + (\ell + 1)\infty^\ell - \ell\infty^{\ell-1}
\end{align*}
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Differential counting polynomial

Example

\[
\left\{ (u(t), v(t)) \in \mathbb{C}[[t - t_0]]^2 \mid v(t) \frac{\partial}{\partial t} u(t) - u(t) - \frac{1}{t} = 0, \frac{\partial^2}{\partial t^2} v(t) = 0 \right\}
\]

has differential counting polynomial \( \infty^3 - \infty^2 + \infty - \aleph_0 \) for \( t_0 \neq 0 \). All solutions are analytical.
Example

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- Only unique if no \( \aleph_0 \) appears (\( \infty - \aleph_0 \triangleq \infty - \aleph_0 + 1 \)).
Differential counting polynomial

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- Not algorithmic.
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Theorem

The degree (in \( \infty \)) of the differential counting polynomial of a characterizable differential ideal \( I \) is its dimension polynomial \( \omega_I(\ell) \).
Theorem

If the system of differential equations is given by a simple system $S$ without inequations then the differential counting polynomial is

$$
\left( \prod_{p \in S} \deg(p) \right) \cdot \infty^{\omega_I(\ell)},
$$

where $I$ is the characterizable differential ideal of $S$. 

The differential counting polynomial of

$$
\frac{\partial^2}{\partial x^2} u(t, x) + \frac{\partial}{\partial t} u(t, x) + 2 u(t, x) \frac{\partial}{\partial x} u(t, x) = 0,
$$

the Burgers' equation, is $\infty^{2\ell + 1}$.

Similarly: semilinear examples "from nature" (Ricatti, Navier-Stokes, Korteweg-de-Vries, Klein-Gordon, ...).
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$$\infty^{2\ell+1}.$$

Similarly: semilinear examples “from nature” (Ricatti, Navier-Stokes, Korteweg-de-Vries, Klein-Gordon, . . . ).
Diff. counting polynomial: comparing solution sets

Theorem

Let \( P \subseteq Q \) sets of power series solutions.

1. If neither \( c(P) \) nor \( c(Q) \) contain the indeterminate \( \mathfrak{A}_0 \), then \( P = Q \) iff \( c(P) = c(Q) \).

2. In general one can estimate \( \mathfrak{A}_0 \) to show \( P \neq Q \).
**Theorem**

Let \( P \subseteq Q \) sets of power series solutions.

1. If neither \( c(P) \) nor \( c(Q) \) contain the indeterminate \( \aleph_0 \), then \( P = Q \) iff \( c(P) = c(Q) \).

2. In general one can estimate \( \aleph_0 \) to show \( P \neq Q \).

**Example:**

The ODE \( v(t) \cdot \frac{\partial}{\partial t} u(t) - u(t) = 0 \) has differential counting polynomial \( \infty^{\ell+2} - \infty^{\ell+1} + (\ell + 1)\infty^{\ell} - \ell\infty^{\ell-1} \).

Maple: \( v(t) \) arbitrary and \( u(t) = ce^{\int_0^t \frac{1}{v(h)} \, dh} \).

This set has differential counting polynomial \( \infty^{\ell+2} - \infty^{\ell+1} \).
Thank you for your attention!