Computing Unit Groups of Orders

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The classical Voronoi Algorithm

- Around 1900 Korkine, Zolotareff, and Voronoi developed a reduction theory for quadratic forms.
- The aim was to classify the densest lattice sphere packings in \( n \)-dimensional Euclidean space.
- Lattice \( L = \mathbb{Z}^{1 \times n} \), Euclidean structure on \( L \) given by some positive definite \( F \in \mathbb{R}^{n \times n}_{\text{sym}}, (x, y) = xFy^\text{tr} \).
- Voronoi described an algorithm to find all local maxima of the density function on the space of all \( n \)-dimensional positive definite \( F \).
- They are perfect forms (as will be defined below).
- There are only finitely many perfect forms up to the action of \( \text{GL}_n(\mathbb{Z}) \), the unit group of the order \( \mathbb{Z}^{n \times n} \).
- Later, Voronoi’s algorithm has been used to compute generators and relations for \( \text{GL}_n(\mathbb{Z}) \) but also its integral homology groups.
- It has been generalised to other situations: compute integral normalizer, the automorphism group of hyperbolic lattices and
- more general unit groups of orders.
Unit groups of orders

- A separable \( \mathbb{Q} \)-algebra, so \( A \cong \bigoplus_{i=1}^{s} D_{i}^{n_{i} \times n_{i}} \), is a direct sum of matrix rings over division algebras.

- An order \( \Lambda \) in \( A \) is a subring that is finitely generated as a \( \mathbb{Z} \)-module and such that \( \langle \Lambda \rangle_{\mathbb{Q}} = A \).

- Its unit group is \( \Lambda^* := \{ u \in \Lambda \mid \exists v \in \Lambda, uv = 1 \} \).

- Know in general: \( \Lambda^* \) is finitely generated.

- Example: \( A = K \) a number field, \( \Lambda = O_{K} \), its ring of integers. Then Dirichlet’s unit theorem says that \( \Lambda^* \cong \mu_{K} \times \mathbb{Z}^{r+s-1} \).

- Example: \( \Lambda = \langle 1, i, j, ij \rangle_{\mathbb{Z}} \) with \( i^2 = j^2 = (ij)^2 = -1 \). Then \( \Lambda^* = \langle i, j \rangle \) the quaternion group of order 8.

- Example: \( A = \mathbb{Q}G \) for some finite group \( G \), \( \Lambda = \mathbb{Z}G \).

- Example: \( A \) a division algebra with \( \dim_{\mathbb{Z}(A)}(A) = d^2 > 4 \). Not much known about the structure of \( \Lambda^* \).

- Voronoi’s algorithm may be used to compute generators and relations for \( \Lambda^* \) and to solve the word problem.

- Seems to be practical for “small” \( A \) and for \( d = 3 \).
The classical Voronoi Algorithm
Korkine, Zolotareff, Voronoi, ∼ 1900.

**Definition**

- $\mathcal{V} := \{ X \in \mathbb{R}^{n \times n} \mid X = X^{tr} \}$ space of symmetric matrices
- $\sigma : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$, $\sigma(A, B) := \text{trace}(AB)$ Euclidean inner product on $\mathcal{V}$.
- for $F \in \mathcal{V}$, $x \in \mathbb{R}^{1 \times n}$ define $F[x] := xFx^{tr} = \sigma(F, x^{tr}x)$
- $\mathcal{V}^{>0} := \{ F \in \mathcal{V} \mid F \text{ positive definite } \}$
- for $F \in \mathcal{V}^{>0}$ define the minimum $\mu(F) := \min\{ F[x] : 0 \neq x \in \mathbb{Z}^{1 \times n} \}$ and $\mathcal{M}(F) := \{ x \in \mathbb{Z}^{1 \times n} \mid F[x] = \mu(F) \}$
- $\text{Vor}(F) := \{ \sum_{x \in \mathcal{M}(F)} a_xx^{tr}x \mid a_x \geq 0 \}$ the Voronoi domain
- $F$ is called **perfect** $\iff \dim(\text{Vor}(F)) = \dim(\mathcal{V}) = \frac{n(n+1)}{2}$.

**Remark**

$\text{GL}_n(\mathbb{Z})$ acts on $\mathcal{V}^{>0}$ by $(F, g) \mapsto g^{-1}Fg^{-tr}$. Then

$$
\mathcal{M}(g^{-1}Fg^{-tr}) = \{ xg \mid x \in \mathcal{M}(F) \}
$$

$$
\text{Vor}(g^{-1}Fg^{-tr}) = g^{tr}\text{Vor}(F)g
$$
The classical Voronoi Algorithm
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**Theorem (Voronoi)**

(a) $\mathcal{T} := \{ \text{Vor}(F) \mid F \in \mathcal{V}^{>0}, \text{ perfect } \}$ forms a face to face tessellation of $\mathcal{V}^{\geq0}$.
(b) $\text{GL}_n(\mathbb{Z})$ acts on $\mathcal{T}$ with finitely many orbits that may be computed algorithmically.
Example, generators for $\text{GL}_2(\mathbb{Z})$

- $n = 2$, $\dim(V) = 3$, $\dim(V^>\mathbb{R}^>) = 2$
- compute in affine section of the projective space
- $\mathcal{A}^\geq = \{ F \in V^\geq \mid \text{trace}(F) = 1 \}$
- $F_0 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\mu(F_0) = 2$, $\mathcal{M}(F_0) = \{ \pm(1, 0), \pm(0, 1), \pm(1, 1) \}$
- $\mathcal{A}^\geq \cap \text{Vor}(F_0) = \text{conv}(a = \begin{pmatrix} 10 \\ 00 \end{pmatrix}, b = \begin{pmatrix} 00 \\ 01 \end{pmatrix}, c = \frac{1}{2} \begin{pmatrix} 11 \\ 11 \end{pmatrix})$
Example, generators for $GL_2(\mathbb{Z})$

- $n = 2$, $\dim(V) = 3$, $\dim(V^{>0}/\mathbb{R}_{>0}) = 2$
- compute in affine section of the projective space
- $A^{\geq 0} = \{ F \in V^{\geq 0} \mid \text{trace}(F) = 1 \}$
- $F_0 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\mu(F_0) = 2$, $M(F_0) = \{ \pm(1, 0), \pm(0, 1), \pm(1, 1) \}$
- $A^{\geq 0} \cap \text{Vor}(F_0) = \text{conv}(a = \begin{pmatrix} 10 \\ 00 \end{pmatrix}, b = \begin{pmatrix} 00 \\ 01 \end{pmatrix}, c = \frac{1}{2} \begin{pmatrix} 11 \\ 11 \end{pmatrix})$
Example, generators for $\text{GL}_2(\mathbb{Z})$

- Compute neighbor: $F_1 \in \mathcal{V}^>^0$ so that $\text{Vor}(F_1) = \text{conv}(a, b, c')$.
- Linear equation on $F_1$: $\text{trace}(F_1a) = \text{trace}(F_1b) = 2$ and $\text{trace}(F_1c) > 2$.
- So $F_1 = F_0 + sX$ where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ generates $\langle a, b \rangle^\perp$.
- For $s = 2$ the matrix $F_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has again 6 minimal vectors
- $\mathcal{M}(F_1) = \{ \pm(1, 0), \pm(0, 1), \pm(1, -1) \}$
- $\mathcal{A}^\geq^0 \cap \text{Vor}(F_1) = \text{conv}(a, b, c' := \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix})$
Example, generators for $GL_2(\mathbb{Z})$

- $\text{Stab}_{GL_2(\mathbb{Z})}(F_0) = \langle g = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$
- $(a, b) \cdot g = (b, c), (b, c) \cdot g = (c, a)$
- Compute isometry $t = \text{diag}(1, -1) \in GL_2(\mathbb{Z})$, so $t^{-1}F_0t^{-tr} = F_1$.
- Then $GL_2(\mathbb{Z}) = \langle g, h, t \rangle$. 
\( \text{GL}_2(\mathbb{Z}) = \langle g, h, t \rangle \).

- \( \text{Stab}_{\text{GL}_2(\mathbb{Z})}(F_0) = \langle g = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \)
- \((a, b) \cdot g = (b, c), (b, c) \cdot g = (c, a)\)
- Compute isometry \( t = \text{diag}(1, -1) \in \text{GL}_2(\mathbb{Z}) \), so \( t^{-1} F_0 t^{-tr} = F_1 \).
Variations of Voronoi’s algorithm

- Many authors used this algorithm to compute integral homology groups of $\text{SL}_n(\mathbb{Z})$ and related groups, as developed C. Soulé in 1978.

- **Max Köcher** developed a general Voronoi Theory for pairs of dual cones in the 1950s. 
  \[ \sigma : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{R} \] 
  non degenerate and positive on the cones $\mathcal{V}_1^> \times \mathcal{V}_2^>$. 
  discrete admissible set $D \subset \mathcal{V}_2^>$ used to define minimal vectors and perfection for $F \in \mathcal{V}_1^>$ and $\text{Vor}_D(F) \subset \mathcal{V}_2^>$.

- **J. Opgenorth** (2001) used Köcher’s theory to compute the integral normalizer $N_{\text{GL}_n(\mathbb{Z})}(G)$ for a finite unimodular group $G$.

- **M. Mertens** (Masterthesis, 2012) applied Köcher’s theory to compute automorphism groups of hyperbolic lattices.

- **This talk** will explain how to apply it to obtain **generators and relations** for **unit group of orders** in semi-simple rational algebras and an algorithm to solve the **word problem** in these generators.
Orders in semi-simple rational algebras.

The positive cone

- $K$ some rational division algebra, $A = K^{n \times n}$
- $A_R := A \otimes Q \mathbb{R}$ semi-simple real algebra
- so $A_R$ is isomorphic to a direct sum of matrix rings over of $\mathbb{H}$, $\mathbb{R}$ or $\mathbb{C}$.
- $A_R$ carries a “canonical” involution $^\dagger$ (depending on the choice of the isomorphism) that we use to define symmetric elements:
- $V := \text{Sym}(A_R) := \{ F \in A_R \mid F^\dagger = F \}$
- $\sigma(F_1, F_2) := \text{trace}(F_1 F_2)$ defines a Euclidean inner product on $V$.
- In general the involution $^\dagger$ will not fix the set $A$.

The simple $A$-module.

- Let $V = K^{1 \times n}$ denote the simple right $A$-module, $V_R = V \otimes Q \mathbb{R}$.
- For $x \in V$ we have $x^\dagger x \in V$.
- $F \in V$ is called positive if
  \[ F[x] := \sigma(F, x^\dagger x) > 0 \] for all $0 \neq x \in V_R$. 

Minimal vectors.

The discrete admissible set

- $\mathcal{O}$ maximal order in $K$, $L$ some $\mathcal{O}$-lattice in the simple $A$-module $V$
- $\Lambda := \text{End}_\mathcal{O}(L)$ is a maximal order in $A$ with unit group $\Lambda^* := \text{GL}(L) = \{a \in A \mid aL = L\}$.

$L$-minimal vectors

Let $F \in \mathcal{V}^{>0}$.

- $\mu(F) := \mu_L(F) = \min\{F[\ell] \mid 0 \neq \ell \in L\}$ the $L$-minimum of $F$.
- $\mathcal{M}_L(F) := \{\ell \in L \mid F[\ell] = \mu_L(F)\}$ the finite set of $L$-minimal vectors.
- $\text{Vor}_L(F) := \{\sum_{x \in \mathcal{M}_L(F)} a_x x^\dagger x \mid a_x \geq 0\} \subset \mathcal{V}^{\geq0}$ Voronoi domain of $F$.
- $F$ is called $L$-perfect $\iff \dim(\text{Vor}_L(F)) = \dim(\mathcal{V})$.

Theorem

$\mathcal{T} := \{\text{Vor}_L(F) \mid F \in \mathcal{V}^{>0}, \text{ L-perfect} \}$ forms a face to face tessellation of $\mathcal{V}^{\geq0}$. $
\Lambda^*$ acts on $\mathcal{T}$ with finitely many orbits.
Generators for $\Lambda^*$

- Compute $\mathcal{R} := \{F_1, \ldots, F_s\}$ set of representatives of $\Lambda^*$-orbits on the $L$-perfect forms, such that their Voronoi-graph is connected.
- For all neighbors $F$ of one of these $F_i$ (so $\text{Vor}(F) \cap \text{Vor}(F_i)$ has codimension 1) compute some $g_F \in \Lambda^*$ such that $g_F \cdot F \in \mathcal{R}$.
- Then $\Lambda^* = \langle \text{Aut}(F_i), g_F \mid F_i \in \mathcal{R}, F \text{ neighbor of some } F_j \in \mathcal{R} \rangle$.

so here $\Lambda^* = \langle \text{Aut}(F_1), \text{Aut}(F_2), \text{Aut}(F_3), a, b, c, d, e, f \rangle$. 
Example $Q_{2,3}$.

- Take the rational quaternion algebra ramified at 2 and 3,
  
  $Q_{2,3} = \langle i, j \mid i^2 = 2, j^2 = 3, ij = -ji \rangle = \langle \text{diag}(\sqrt{2}, -\sqrt{2}), \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \rangle$

  Maximal order $\Lambda = \langle 1, i, \frac{1}{2}(1 + i + ij), \frac{1}{2}(j + ij) \rangle$

- $V = A = Q_{2,3}$, $A_{\mathbb{R}} = \mathbb{R}^{2 \times 2}$, $L = \Lambda$

- Embed $A$ into $A_{\mathbb{R}}$ using the maximal subfield $\mathbb{Q}[\sqrt{2}]$.

- Get three perfect forms:
  
  - $F_1 = \begin{pmatrix} 1 & 2 - \sqrt{2} \\ 2 - \sqrt{2} & 1 \end{pmatrix}$, $F_2 = \begin{pmatrix} 6 - 3\sqrt{2} & 2 \\ 2 & 2 + \sqrt{2} \end{pmatrix}$
  
  - $F_3 = \text{diag}(-3\sqrt{2} + 9, 3\sqrt{2} + 5)$
The tessellation for $\mathbb{Q}_{2,3} \hookrightarrow \mathbb{Q}[\sqrt{2}]^{2\times2}$. 

![Diagram of the tessellation]
\( \Lambda^*/\langle \pm 1 \rangle = \langle a, b, t \mid a^3, b^2, atbt \rangle \)
Easy solution of the word problem
Easy solution of the word problem
Easy solution of the word problem
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The tesselation for $\mathbb{Q}_{2,3} \hookrightarrow \mathbb{Q}[\sqrt{3}]^{2\times2}$. 
Conclusion

- Algorithm works quite well for indefinite quaternion algebras over the rationals
- Obtain presentation and algorithm to solve the word problem
- For \( Q_{19,37} \) our algorithm computes the presentation within 5 minutes (288 perfect forms, 88 generators) whereas the Magma implementation “FuchsianGroup” does not return a result after four hours
- Reasonably fast for quaternion algebras with imaginary quadratic center or matrix rings of degree 2 over imaginary quadratic fields
- For the rational division algebra of degree 3 ramified at 2 and 3 compute presentation of \( \Lambda^* \), 431 perfect forms, 50 generators in about 10 minutes.
- Quaternion algebra with center \( \mathbb{Q}[\zeta_5] \): \( > 40,000 \) perfect forms.
- Masterthesis by Oliver Braun: The tessellation \( T \) can be used to compute the maximal finite subgroups of \( \Lambda^* \).
- Masterthesis by Sebastian Schönnenbeck: Compute integral homology of \( \Lambda^* \).
Calculating maximal finite subgroups
Minimal classes

**Definition**

Let $A, B \in \mathcal{V}^{>0}$. $A$ and $B$ are **minimally equivalent** if $\mathcal{M}_L(A) = \mathcal{M}_L(B)$. $C := \text{Cl}_L(A) = \{X \in \mathcal{V}^{>0} \mid \mathcal{M}_L(X) = \mathcal{M}_L(A)\}$ is the **minimal class** of $A$. In this case $\mathcal{M}_L(C) := \mathcal{M}_L(A)$. Call $C$ **well-rounded** if $\mathcal{M}_L(C)$ contains a $K$-basis of $V = K^{1 \times n}$.

**Remark**

$A \in \mathcal{V}^{>0}$ is $L$-perfect if and only if $\text{Cl}_L(A) = \{\alpha A \mid \alpha \in \mathbb{R}_{>0}\}$. 
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**Remark**
dim$_\mathbb{R}(V) -$ dim$_\mathbb{R}(\langle x^t x \mid x \in M_L(A) \rangle)$, the **perfection corank**, is constant on $\text{Cl}_L(A)$. 
Calculating minimal classes

Theorem

Let $A \in \mathcal{V}^>0$ be $L$-perfect. Any codimension-$k$-face of $\text{Vor}_L(A)$ corresponds to a minimal class of perfection corank $k$, represented by

$$A + \frac{1}{2k} \sum_{i=1}^{k} \rho_i R_i = \frac{1}{k} \sum_{i=1}^{k} \left( A + \frac{\rho_i}{2} R_i \right) \in \mathcal{V}^>0$$

with facet vectors $R_i$ and $\rho_i \in \mathbb{R}_{>0}$ such that $A + \rho_i R_i$ is a perfect neighbour of $A$ (and the codimension-$k$-face in question is the intersection is the intersection of the facets with facet vectors $R_i$).
Calculating minimal classes

**Theorem**

Let \( A \in \mathcal{V}^{>0} \) be \( L \)-perfect. Any codimension-\( k \)-face of \( \text{Vor}_L(A) \) corresponds to a minimal class of perfection corank \( k \), represented by

\[
A + \frac{1}{2k} \sum_{i=1}^{k} \rho_i R_i = \frac{1}{k} \sum_{i=1}^{k} \left( A + \frac{\rho_i}{2} R_i \right) \in \mathcal{V}^{>0}
\]

with facet vectors \( R_i \) and \( \rho_i \in \mathbb{R}_{>0} \) such that \( A + \rho_i R_i \) is a perfect neighbour of \( A \) (and the codimension-\( k \)-face in question is the intersection is the intersection of the facets with facet vectors \( R_i \)).

**Example: The minimal classes in dimension 2 over \( \mathbb{Z} \)**

\( F_0 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \), \( \mathcal{M}(F_0) = \{ \pm(1, 0), \pm (0, 1), \pm (1, 1) \} \)

\( \mathcal{A}^{\geq 0} \cap \text{Vor}(F_0) = \text{conv} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \)

Facet vectors \( R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( R_2 = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} \), \( R_3 = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \)

All perfect neighbours of \( F_0 \) are given by \( F_0 + 2R_i \), so \( \rho_i = 2 \) for all \( 1 \leq i \leq 3 \).
Now consider the dual of the tesselation of $\mathcal{T} = \{ \text{Vor}(F) \mid F \in \mathcal{V}^> \text{ perfect} \}$. 
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- The well-rounded classes have perfection corank 0 or 1. The corank 0 class is the perfect class of $F_0$.
- The corank 1 classes are represented by
  
  \[
  F_0 + R_1 = \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]

  \[
  F_0 + R_2 = \begin{pmatrix}
4 & -2 \\
-2 & 2
\end{pmatrix}
\]

  \[
  F_0 + R_3 = \begin{pmatrix}
2 & -2 \\
-2 & 4
\end{pmatrix}
\]

- The minimal classes represented by these three matrices are in the same orbit under $\text{GL}_2(\mathbb{Z})$. This is easily checked for well-rounded minimal classes using a theorem by A.-M. Bergé.

- The corank 2 class is represented by $\frac{1}{2} (2F_0 + R_1 + R_2) = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$. 

- The corank 3 class is represented by $\frac{1}{2} (2F_0 + R_1 + R_2 + R_3) = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$.
Maximal finite subgroups

**Theorem (Coulangeon, Nebe (2013))**

$G \leq GL(L)$ maximal finite $\implies G = \text{Aut}_L(C)$, where $C$ is a well-rounded minimal class, such that $\dim(C \cap F(G)) = 1$. 

$F(G) := \{ A \in V \mid A[g] = A \ \forall \ g \in G\}$

**Remark**

This theorem yields a finite set of finite subgroups of $GL(L)$, containing a set of representatives of conjugacy classes of maximal finite subgroups of $GL(L)$. 

There are algorithmic methods to check if a finite subgroup is maximal finite and whether two maximal finite subgroups are conjugate. Therefore in the previous example, we obtain two conjugacy classes of maximal finite subgroups: 

- The stabilizer of the perfect form $F_0$, which is isomorphic to $D_{12}$. 
- The stabilizer of the corank 1 class, which is isomorphic to $D_8$.

These groups are indeed maximal finite.
Maximal finite subgroups

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  The stabilizer of the perfect form \( F_0 \), which is isomorphic to \( D_{12} \).
  The stabilizer of the corank 1 class, which is isomorphic to \( D_8 \).
- These groups are indeed maximal finite.
Example: \( \mathbb{Q}(\sqrt{-6}) \)

\[ O = \mathbb{Z}[\sqrt{-6}], \ L_0 = O \oplus O, \ L_1 = O \oplus p, \text{ where } p \mid (2). \]

Well-rounded minimal classes for \( \mathbb{Q}(\sqrt{-6}) \)

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<th>( G = \text{Aut}_L(C) )</th>
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The cell decomposition of $\mathcal{V}^>0$

**Minimal classes**

For $F \in \mathcal{V}^>0$ define $C_l(F) := \{F' \in \mathcal{V}^>0 \mid M_L(F') = M_L(F)\}$ the minimal class corresponding to $F$. 

The decomposition $\mathcal{V}^>0$ decomposes into the disjoint union of all minimal classes.

Properties of this decomposition:

▶ Partial ordering on the minimal classes: $C \preceq C' \iff M_L(C) \subseteq M_L(C')$.

▶ Each minimal class is a convex set in $\mathcal{V}$.

▶ The decomposition as well as the partial ordering are compatible with the $\Lambda^*$-action.

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- $F \in \mathcal{V}^>^0$ is called *well-rounded*, if $\mathcal{M}_L(F')$ contains a $K$-Basis of $K^n$. 

}$\begin{align*}
\mathcal{V}^>^0, wr &:= 
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Properties of the well-rounded retract

In $\mathcal{V}_{1}^{0,wr}$ we have:

- There are only finitely many $\Lambda^*$-orbits in any dimension and every occurring stabiliser is finite.
- The topological closure of each cell is a polytope.
- $\mathcal{V}_{1}^{0,wr}$ is a retract of $\mathcal{V}_{1}^{0}$, especially we have that the cellular chain complex is again acyclic and $H_0 \cong \mathbb{Z}$ (A. Ash, 1984).

Summary

- The group $\Lambda^*$ acts on the space of positive definite forms.
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