

# Modular Harish-Chandra series

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$\rightsquigarrow$  Algebraic group techniques (highest weight theory, ...).

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$\rightsquigarrow$  Harish-Chandra theory: Inductive approach.

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[Above references for general case  $\text{char}(k) = \ell \neq p$ ; for  $\ell = 0$  see book by CURTIS-REINER]

# Complete results

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