Modular Harish-Chandra series

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$\rightsquigarrow$ Algebraic group techniques (highest weight theory, ...).

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$\rightsquigarrow$ Harish-Chandra theory: Inductive approach.

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[Above references for general case $\mathrm{char}(k) = \ell \neq p$; for $\ell = 0$ see book by Curtis–Reiner]

Complete results

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Now $G = G(q)$ general linear, unitary, symplectic or orthogonal.

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- Directed edge $\lambda \to \mu$ if:
  - $|\lambda| = 2m+\delta$ and $|\mu| = |\lambda| + 2$ for some $m \geqslant 0$;
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Problem solved for $\mathrm{GL}_n(q)$ [Dipper–James 1980/90s, Dipper–Du 1997]

Fix $\delta \in \{0,1\}$ and define $G_m := \mathrm{GU}_{2m+\delta}(q)$ for $m = 0, 1, 2, \ldots$.

- Usual Mackey system $\mathcal{L}_{G_m} = \{(L,P) \mid P \text{ parabolic}, L = P/U_P\}$.
  Here, $L \cong G_{m'} \times$ direct product of $\mathrm{GL}_{n_i}(q^2)$ for various n_i; $m = m' + \sum_i n_i$.
- Gerber–Hiss–Jacon 2015: "Weak" Mackey system
  - $\mathcal{L}'_{G_m} = \{(L,P) \in \mathcal{L}_{G_m} \mid L = P/U_P$ "pure", i.e., all $n_i = 1\}$.
  - Obtain weak HC-series, weakly cuspidal modules.

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Then: Sources $\leftrightarrow$ weakly cuspidal modules; paths $\leftrightarrow$ weak HC-series.

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  - to prove analogous results for symplectic and orthogonal groups (by-passing missing triangularity conjecture in these cases).