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# Well-posedness of Non-autonomous Linear Evolution Equations in Uniformly Convex Spaces

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## Abstract

This paper addresses the problem of wellposedness of non-autonomous linear evolution equations  $\dot{x} = A(t)x$  in uniformly convex Banach spaces. We assume that  $A(t) : D \subset X \rightarrow X$ , for each  $t$  is the generator of a quasi-contractive  $C_0$ -group where the domain  $D$  and the growth exponent are independent of  $t$ . Well-posedness holds provided that  $t \mapsto A(t)y$  is Lipschitz for all  $y \in D$ . Hölder continuity of degree  $\alpha < 1$  is not sufficient and the assumption of uniform convexity cannot be dropped.

## 1 Introduction

In the literature the existence of the propagator (evolution system) for the non-autonomous Schrödinger equation is often discussed within the more general context of abstract non-autonomous linear evolution equation

$$\dot{x} = A(t)x, \quad x(s) = y \tag{1}$$

in some Banach space  $X$  where  $A(t) : D(A(t)) \subset X \rightarrow X$  for each  $t \in [0, T]$  is the generator of a strongly continuous semigroup. On the level of proofs this approach involves serious technical difficulties that are associated with the lack of structure of general Banach spaces and the non-reversibility of the dynamics given by a semigroup. The prize for the solution of these problems is paid in terms of the regularity required of  $t \mapsto A(t)$  [6, 7, 2, 12, 9, 4].

In the present paper, which is motivated by the Schrödinger equation, the evolution problem (1) is discussed in a more restrictive setting, which does not have the drawbacks mentioned above. In this setting  $X$  is a uniformly convex Banach space and  $A(t)$ , for each  $t \in [0, T]$  is the generator of a strongly continuous *group* rather than a semigroup. We assume, moreover, that this group is quasi-contractive with a growth exponent that is independent of  $t$  and that the domain  $D = D(A(t))$  is independent of  $t$  as well. Our main result, in the simplest form, establishes the existence of a unique evolution system  $U(t, s)$  provided

$$t \mapsto A(t)y \tag{2}$$

is Lipschitz for all  $y \in D$ . It follows that  $t \mapsto x(t) = U(t, s)y$  is the unique continuously differentiable solution of (1) and that it depends continuously on the initial data  $s$  and  $y$  (well-posedness). We give examples showing that Hölder continuity of the map (2) is not sufficient and that Lipschitz continuity is not sufficient anymore if the assumption

of uniform convexity is dropped. This means in particular that well-posedness of the non-autonomous Schrödinger equation - the Equation (1) if  $X$  is a Hilbert space and  $A(t)^* = -A(t)$  - requires less regularity than well-posedness of (1) in the general Banach space setting.

The well-posedness of (1) in uniformly convex Banach spaces was previously studied by Kato [6]. Our result described above could be derived from the Theorem 5.2 combined with the information from the Remark 5.3 in [6]. See Theorem 3.2 of [13] for a Hilbert space version of this result of Kato's. The more general version of our main result, the Theorem 2.1 below, combined with the Remark 2.3, does not follow from previous work and our counterexamples are new as well. Moreover, the present paper shows that the essence of Kato's work in the uniformly convex case can be summarized in a short and simple proof that requires nothing but basic functional analysis and a rudimentary knowledge of semigroup theory. This we consider the main message of our paper.

## 2 Preparations, Results and Examples

Let  $X$  be a complex Banach space and let  $A(t) : D \subset X \rightarrow X$ ,  $t \in [0, T]$  be a family of closed linear operators with a time-independent dense domain  $D \subset X$ . A two-parameter family of linear operators  $U(t, s) \in \mathcal{L}(X)$ , will be called an *evolution system for  $A(t)$*  on  $D$  if the following conditions are satisfied:

- (i)  $U(t, s)D \subset D$  and the map  $t \mapsto U(t, s)y$  on  $[0, T]$  is a continuously differentiable solution of (1) for any  $y \in D$  and  $s \in [0, T]$ .
- (ii)  $U(s, s) = 1$  and  $U(t, r)U(r, s) = U(t, s)$  for all  $s, r, t \in [0, T]$ .
- (iii)  $(t, s) \mapsto U(t, s)$  strongly continuous on  $[0, T] \times [0, T]$ .

Any two-parameter family of linear operators  $U(t, s) \in \mathcal{L}(X)$  satisfying (ii) and (iii) is called an *evolution system*.

Existence of an evolution system  $U(t, s)$  with the properties analogous to (i)-(iii) on the triangle  $0 \leq s \leq t \leq T$  is equivalent to well-posedness in the classical sense of  $C^1$ -solutions [3], Proposition VI.9.3. Our assumptions on  $A(t)$  in Theorem 2.1 will allow us to construct  $U(t, s)$  on the entire square  $[0, T] \times [0, T]$  and this is essential for our proof.

For the reader's convenience we recall that a Banach space  $X$  is called uniformly convex if, given  $\varepsilon > 0$  there exists  $\delta > 0$  such that any pair of normalized vectors  $x, y \in X$  with  $\|(x + y)/2\| > 1 - \delta$  satisfies  $\|x - y\| < \varepsilon$ . Every Hilbert space and every  $L^p(\mathbb{R}^n)$  with  $1 < p < \infty$  is uniformly convex. Uniformly convex Banach spaces are reflexive (Milman) and uniform convexity implies that weak convergence  $x_n \rightharpoonup x$  turns into strong convergence as soon as  $\|x_n\| \rightarrow \|x\|$ .

As a final preparation we recall from [9, 3] that the norm of every strongly continuous semigroup  $e^{At}$ ,  $t \geq 0$ , satisfies a bound of the form  $\|e^{At}\| \leq Me^{\omega t}$ . It follows that  $\sigma(A) \subset \{\operatorname{Re} z \leq \omega\}$ . If  $M = 1$  then the semigroup is called *quasi-contractive*.

**Theorem 2.1.** *Let  $X$  be a uniformly convex Banach space and let  $A(t) : D \subset X \rightarrow X$ , for each  $t \in [0, T]$  be the generator of a strongly continuous group with*

$$\|e^{A(t)s}\| \leq e^{\omega|s|}, \quad s \in \mathbb{R}, \quad (3)$$

where  $\omega$  and the domain  $D$  are independent of  $t$ . Suppose that  $t \mapsto A(t)y$  is Lipschitz for all  $y \in D$ . Then there exists a unique evolution systems  $U(t, s)$  for  $A(t)$  on  $D$ .

*Remark 2.2.* This theorem is false if the assumption of uniform convexity is dropped (Example 1), and moreover, even if  $X$  is a Hilbert space and  $A(t)$  is skew-selfadjoint the Lipschitz continuity cannot be replaced by Hölder continuity of some degree  $\alpha < 1$  (Example 2).

*Proof.* Let  $\underline{A}(t) := A(t) - (\omega + 1)$ . We define  $\|y\|_t := \|\underline{A}(t)y\|$  which is equivalent to the graph norm of  $\underline{A}(t)$ . Hence  $Y_t = (Y, \|\cdot\|_t)$  is a Banach space. Like  $X$  the space  $Y_t$  is uniformly convex as can be easily verified using the definition of uniform convexity given above.

In the special case where  $X$  is a Hilbert space and  $A(t)$  is skew-selfadjoint it follows that  $\|y\|_t^2 = \|A(t)y\|^2 + \|y\|^2$  and hence  $Y_t$  is a Hilbert space too.

Let  $Y = Y_0$  and note that the strong Lipschitz continuity of  $t \mapsto A(t)$  is equivalent to the Lipschitz continuity of  $t \mapsto A(t) \in \mathcal{L}(Y, X)$  by the principle of uniform boundedness. This means that

$$\|A(t) - A(s)\|_{Y, X} \leq L|t - s| \quad (4)$$

for some  $L$  and all  $s, t \in [0, T]$ .

**Step 1:** There exists a constant  $c$  such that for all  $s, t \in [0, T]$  and all  $y \in D$ ,

$$\|y\|_t \leq e^{c|t-s|} \|y\|_s.$$

*Proof.* By the continuity of  $t \mapsto \underline{A}(t) \in \mathcal{L}(Y, X)$ , the map  $t \mapsto \underline{A}(t)^{-1} \in \mathcal{L}(X, Y)$  is continuous and hence  $C := \sup_{s \in I} \|\underline{A}(s)^{-1}\|_{X, Y} < \infty$ . In view of  $\|y\|_t \leq \|\underline{A}(t)\underline{A}(s)^{-1}\| \|y\|_s$ , the Step 1 with  $c = CL$  follows from

$$\begin{aligned} \|\underline{A}(t)\underline{A}(s)^{-1}\| &= \|1 + (\underline{A}(t) - \underline{A}(s))\underline{A}(s)^{-1}\| \\ &\leq 1 + \|\underline{A}(t) - \underline{A}(s)\|_{Y, X} \|\underline{A}(s)^{-1}\|_{X, Y} \leq 1 + CL|t - s| \leq e^{CL|t-s|}. \end{aligned}$$

We now choose a sequence of partitions  $\pi_n$  of  $[0, T]$  with the property that the mesh size of  $\pi_n$  vanishes in the limit  $n \rightarrow \infty$ . Given  $t \in [0, T]$  and  $n \in \mathbb{N}$  we use  $t_n$  to denote the largest element of  $\pi_n$  less or equal to  $t$ . The smallest element of  $\pi_n$  larger than  $t_n$  is denoted  $t_n^+$ , the largest one smaller than  $t_n$  is denoted  $t_n^-$ . We thus have  $t_n^- < t_n < t_n^+$  and

$$t_n \leq t < t_n^+.$$

Note that the points  $t_n$  and  $t_n^\pm$  are functions of both  $t$  and  $n$ . We define  $U_n(t, s)$  for  $t > s$  by

$$U_n(t, s) := e^{A(t_n)(t-t_n)} e^{A(t_n^-)(t_n-t_n^-)} \dots e^{A(s_n)(s_n^+-s)}$$

and  $U_n(s, t) := U_n(t, s)^{-1}$ . Note that  $\|U_n(t, s)\| \leq e^{\omega|t-s|}$  by assumption (3).

**Step 2:** For all  $t > s$ ,  $n \in \mathbb{N}$ , and  $y \in D$ ,

$$\begin{aligned} \|U_n(t, s)y\|_t &\leq e^{(c+\omega)(t-s)+2c(s-s_n)} \|y\|_s \\ \text{and } \|U_n(s, t)y\|_s &\leq e^{(c+\omega)(t-s)+2c(s-s_n)} \|y\|_t. \end{aligned}$$

In particular,  $\|U_n(t, s)\|_{Y, Y} < M$  for all  $s, t \in [0, T]$  and all  $n \in \mathbb{N}$ .

*Proof.* With the help of Step 1 we pass from  $\|\cdot\|_t$  to  $\|\cdot\|_{t_n}$ , then from  $\|\cdot\|_{t_n}$  to  $\|\cdot\|_{t_n^-}$  and so on, where in each step we use that  $e^{A(t)\tau}$  is a quasi-contraction in  $Y_t$  satisfying (3) for any  $t \in I$ . In this way we arrive at

$$\|U_n(t, s)y\|_t \leq e^{c(t-s)+\omega(t-s)}\|y\|_{s_n},$$

which, using Step 1 again, leads to the first of the asserted inequalities. The second one is proved analogously and the uniform bound on  $\|U_n(t, s)\|_{Y, Y}$  now follows from Step 1 and the compactness of  $[0, T]$ .

**Step 3:** For all  $x \in X$ , the limit  $U(t, s)x := \lim_{n \rightarrow \infty} U_n(t, s)x$  exists uniformly in  $s, t \in [0, T]$ . It defines an evolution systems  $U(t, s)$ .

*Proof.* For any  $y \in Y$  the map  $\tau \mapsto U_m(t, \tau)U_n(\tau, s)y$  is piecewise continuously differentiable with possible jumps in the derivative at the partition points from  $\pi_m \cup \pi_n$ . It follows that

$$\begin{aligned} U_n(t, s)y - U_m(t, s)y &= U_m(t, \tau)U_n(\tau, s)y \Big|_{\tau=s}^{\tau=t} \\ &= \int_s^t U_m(t, \tau)(A(\tau_n) - A(\tau_m))U_n(\tau, s)y \, d\tau. \end{aligned}$$

By Step 2 we conclude

$$\|U_n(t, s)y - U_m(t, s)y\| \leq \int_0^1 e^{\omega|t-\tau|} \|A(\tau_n) - A(\tau_m)\|_{Y, X} M \|y\|_Y \, d\tau \rightarrow 0 \quad (n, m \rightarrow \infty)$$

by the continuity of  $\tau \mapsto A(\tau) \in \mathcal{L}(Y, X)$ . The assertion now follows from the density of  $Y \subset X$  and from the uniform boundedness  $\|U_n(t, s)\| \leq e^{\omega|t-s|}$ . It follows that  $(t, s) \mapsto U(t, s)x$  is continuous and the property (ii) of evolution systems is inherited from  $U_n(t, s)$  as well.

**Step 4:**  $U(t, s)D \subset D$  and for all  $y \in D$  and  $s, t \in [0, T]$ ,

$$\|U(t, s)y\|_t \leq e^{(c+\omega)|t-s|}\|y\|_s.$$

*Proof.* Let  $y \in D$ . By Step 2 the sequence  $(U_n(t, s)y)$  is bounded in  $Y_t$  and by Step 3,  $U_n(t, s)y \rightarrow U(t, s)y$  in  $X$ . Since  $Y_t$  is reflexive it follows that  $U(t, s)y \in D$  and that  $U_n(t, s)y \rightarrow U(t, s)y$  weakly in  $Y_t$ . Therefore by the estimates of Step 2,

$$\|U(t, s)y\|_t \leq \liminf_{n \rightarrow \infty} \|U_n(t, s)y\|_t \leq e^{(c+\omega)|t-s|}\|y\|_s.$$

**Step 5:** For all  $y \in D$  the map  $t \mapsto U(t, s)y$  is differentiable in the norm of  $X$  and

$$\frac{d}{dt}U(t, s)y = A(t)U(t, s)y.$$

*Proof.* In view of  $U(t, s)Y \subset Y$  and  $U(t+h, s) = U(t+h, t)U(t, s)$ , see Step 4, it suffices to prove the assertion for  $s = t$ . For any  $y \in D$ ,

$$\begin{aligned} U(t+h, t)y - e^{A(t)h}y &= \lim_{n \rightarrow \infty} e^{A(t)(h+t-\tau)}U_n(\tau, t)y \Big|_{\tau=t}^{\tau=t+h} \\ &= \lim_{n \rightarrow \infty} \int_t^{t+h} e^{A(t)(h+t-\tau)}(A(\tau_n) - A(t))U_n(\tau, s)y \, d\tau. \end{aligned}$$

By Step 2 it thus follows that

$$\begin{aligned} & \left\| \frac{1}{h} (U(t+h, t)y - e^{A(t)h}y) \right\| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{|h|} \left| \int_t^{t+h} e^{\omega|t+h-\tau|} \|A(\tau_n) - A(t)\|_{Y,X} d\tau \right| M \|y\|_Y \\ & = \frac{1}{|h|} \left| \int_t^{t+h} e^{\omega|t+h-\tau|} \|A(\tau) - A(t)\|_{Y,X} d\tau \right| M \|y\|_Y \rightarrow 0 \quad (h \rightarrow 0) \end{aligned}$$

by the continuity of  $\tau \mapsto A(\tau) \in \mathcal{L}(Y, X)$ . The Step 5 now follows from  $(e^{A(t)h}y - y)/h \rightarrow A(t)y$ .

**Step 6:** For all  $y \in D$  the map  $t \mapsto A(t)U(t, s)y$  is continuous in the norm of  $X$ .

*Proof.* By the continuity of  $t \mapsto A(t) \in \mathcal{L}(Y, X)$  it suffices to show that  $t \mapsto U(t, s)y$  is continuous in the norm of  $Y$ . To this end it suffices to show that  $\lim_{h \rightarrow 0} U(t+h, t)y = y$  in the norm of  $Y$  or, equivalently, in the norm of  $Y_t$ . Since  $U(t+h, t)y \rightarrow y$  in  $X$  and since  $h \mapsto U(t+h, t)y$  is bounded in  $Y_t$ , see Step 1 and Step 4, it follows that  $U(t+h, t)y \rightarrow y$  weakly in  $Y_t$ . See the proof of Step 4 for a similar argument. Therefore

$$\begin{aligned} \|y\|_t & \leq \liminf_{h \rightarrow 0} \|U(t+h, t)y\|_t \leq \limsup_{h \rightarrow 0} \|U(t+h, t)y\|_t \\ & = \limsup_{h \rightarrow 0} e^{c|h|} \|U(t+h, t)y\|_{t+h} \leq \limsup_{h \rightarrow 0} e^{(2c+\omega)|h|} \|y\|_t = \|y\|_t. \end{aligned}$$

The weak convergence  $U(t+h, t)y \rightarrow y$  in  $Y_t$  and the convergence of the norms implies norm convergence in  $Y_t$  by the uniform convexity.  $\square$

*Remark 2.3.* 1. The strong Lipschitz continuity of the map  $t \mapsto A(t)$  is a convenient, but not a necessary condition. It may be replaced with the assumption that

$$\|y\|_t \leq e^{|\int_s^t \alpha(\tau) dt|} \|y\|_s \quad (5)$$

with some integrable function  $\alpha : I \rightarrow [0, \infty)$ . In the proof of Theorem 2.1 this assumption replaces the Step 1. Then the subsequent steps continue to hold with corresponding modifications. Condition (5) is satisfied if

$$\|A(t)y - A(s)y\| \leq \left| \int_s^t \alpha(\tau) dt \right| \|y\|_0 \quad (6)$$

with an integrable function  $\alpha$ , possibly different from  $\alpha$  in (5), and for (6) to hold the  $W^{1,1}$ -regularity imposed in [7, Theorem 1] - the main result of [7] - is sufficient. Similarly, the  $W^{1,\infty}$ -condition of Kato [5] implies the strong Lipschitz continuity of  $t \mapsto A(t)$ .

2. If  $A(t)$  was assumed to be the generator of a semigroup, rather than a group, in Theorem 2.1, then the arguments of our proof above still establish existence of a (unique) evolution system  $U(t, s)$  defined on the triangle  $0 \leq s \leq t \leq T$  such that

$$\partial_t^+ U(t, s)y = A(t)U(t, s)y,$$

where  $t \mapsto A(t)U(t, s)y$  is right-continuous and  $\partial_t^+$  denotes the derivative from the right. Moreover,  $\partial_t U(t, s)y = A(t)U(t, s)y$  except possibly for a countable set of  $t$ -values depending on  $y$  and  $s$  (see the proof of Theorem 5.2 of [6]).

3. In the case where the first or higher, suitably defined commutators of the operators  $A(t)$  at distinct times are scalars, the continuity of the map (2) is sufficient for well-posedness [4, 8, 11].
4. In the case where  $X$  is a Hilbert space there are formal similarities between our Theorem 2.1 and the Theorem C.2 of Ammari and Breteaux [1]. In [1] the case of skew-selfadjoint generators with time-independent *form domains* is considered and a notion of well-posedness in a weak sense is established.

In the remainder of this paper we specialize to operator families of the form  $A(t) = A_0 + B(t)$  where  $A_0$  is the generator of a  $C_0$ -group in  $X$ ,  $B(t) \in \mathcal{L}(X)$  and  $t \mapsto B(t)$  is strongly continuous. Suppose that the evolution system  $U(t, s)$  for  $A$  exists. Then, for all  $y \in D(A_0)$ ,

$$U(t, s)y = e^{A_0(t-s)}y + \int_s^t d\tau e^{A_0(t-\tau)}B(\tau)U(\tau, s)y$$

which, by assumption on  $B(t)$ , may be iterated indefinitely into a convergent Dyson series. For the evolution system in the interaction picture we obtain

$$\begin{aligned} e^{-A_0t}U(t, s)e^{A_0s}y &= y + \int_s^t d\tau_1 \tilde{B}(\tau_1)y \\ &+ \int_s^t d\tau_1 \int_s^{\tau_1} d\tau_2 \tilde{B}(\tau_1)\tilde{B}(\tau_2)y + \dots, \end{aligned} \quad (7)$$

where  $\tilde{B}(\tau)y := e^{-A_0\tau}B(\tau)e^{A_0\tau}$ . We have thus shown that the operator family  $U(t, s)$  defined by (7) is the only candidate for the evolution system generated by  $A(t) = A_0 + B(t)$ .

The following theorem is now an immediate corollary of the previous one and Theorem 3.1.1 from [9]. It improves on the Theorem 6.2 of Phillips [10].

**Theorem 2.4.** *Suppose  $X$  is a uniformly convex Banach space and that  $A(t) = A_0 + B(t)$  for  $t \in [0, T]$  where  $A_0 : D \subset X \rightarrow X$  is the generator of a strongly continuous quasi-contractive group in  $X$  and  $B(t) \in \mathcal{L}(X)$ . If  $t \mapsto B(t)y$  is continuous for all  $y \in X$  and Lipschitz for all  $y \in D$ , then there exists a unique evolution system  $U(t, s)$  for  $A$  on  $D$  and  $e^{-A_0t}U(t, s)e^{A_0s}$  is given by the Dyson series (7).*

The following examples show that the assumptions of uniform convexity and Lipschitz continuity in this theorem and hence in the Theorem 2.1 cannot be weakened in an essential way.

**Example 1.** Let  $X = C_0(\mathbb{R})$  be the Banach space of bounded and continuous functions vanishing at infinity - the norm is the usual maximum norm - and let  $e^{A_0t}$  be the strongly continuous group in  $X$  defined by left translations. That is,  $e^{A_0t}x(\xi) = x(\xi + t)$ . We define  $A(t) : D \subset X \rightarrow X$  for  $t \in [0, 1]$  by  $D = D(A_0)$  and

$$A(t) = A_0 + B(t), \quad B(t) = e^{A_0t}B e^{-A_0t}$$

where  $B$  denotes multiplication with the following bounded function  $f : \mathbb{R} \rightarrow [0, 1]$ : we choose  $f(\xi) = 0$  for  $\xi \leq 0$ ,  $f(\xi) = \xi$  for  $\xi \in [0, 1]$  and  $f(\xi) = 1$  for  $\xi \geq 1$ . Then  $B(t)$  is multiplication with the function  $f(\xi + t)$  and from the fact that  $f$  is Lipschitz it is easy

to check that  $t \mapsto B(t)$  is strongly Lipschitz. If the evolution system  $U(t, s)$  existed, then it would be given by the Dyson series (7) and since  $\tilde{B}(t) = B$  it would follow that

$$U(t, 0) = e^{A_0 t} e^{Bt}. \quad (8)$$

Since  $D = D(A_0)$  is left invariant by  $e^{A_0 t}$  it would follow that  $e^{Bt} D(A_0) \subset D(A_0)$ . But  $D(A_0) = \{y \in C^1(\mathbb{R}) \mid y, y' \in X\}$  and the operator  $e^{Bt}$  acts as multiplication with the non-differentiable function  $e^{f(\xi)t}$ . Hence  $e^{Bt} D(A_0) \not\subset D(A_0)$  and we have a contradiction. Therefore an evolution system  $U$  for  $A$  on  $D$  cannot exist.

**Example 2.** For this example we adopt all elements of Example 1 with two exceptions: now  $X = L^2(\mathbb{R})$  and  $f$  denotes multiplication with a bounded function  $f = ig_\alpha$ , where  $g_\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is nowhere differentiable but Hölder continuous of degree  $\alpha$  for given  $\alpha < 1$ . An example of such a function is the Weierstraß function

$$g_\alpha(\xi) = \sum_{n=1}^{\infty} 2^{-n\alpha} \cos(2^n \xi),$$

see, e.g., [14], Theorem II.4.9. It easily follows that  $t \mapsto B(t)x$  is Hölder continuous of degree  $\alpha$  for all  $x \in X$ . As in Example 1 we argue that  $e^{Bt} D(A_0) \subset D(A_0)$  if the evolution system  $U$  for  $A$  exists. But  $e^{Bt}$  acts by multiplication with  $\xi \mapsto e^{f(\xi)t}$ , which is nowhere differentiable, and  $D(A_0) = H^1(\mathbb{R})$  whose elements are differentiable almost everywhere. We have a contradiction and hence an evolution system  $U$  for  $A$  on  $D$  cannot exist. Note that  $A(t)$  is skew-selfadjoint in this example.

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