On the Prime Graph of the Unit Group of Integral Group Rings of Finite Groups II

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1. Introduction

Let $G$ be a group. The prime graph $\Pi(G)$ is defined as follows. The vertices of $\Pi(G)$ are the primes $p$ for which $G$ has an element of order $p$. Two different vertices $p$ and $q$ are joined by an edge provided there is an element of $G$ of order $pq$.

The integral group ring of $G$ is denoted by $\mathbb{Z}G$ and $V(\mathbb{Z}G)$ denotes the subgroup of the unit group $U(\mathbb{Z}G)$ consisting of all units with augmentation 1. The object of this article, which is a continuation of [14], is the prime graph of $V(\mathbb{Z}G)$ in the case when $G$ is a finite group.

The question is whether the prime graph of $V(\mathbb{Z}G)$ coincides with that of $G$ [13, Problem 21]. This question, known as the “Prime Graph Question” (PQ) may be regarded as a weak version of the first Zassenhaus conjecture ZC–1 which says that each torsion unit of $V(\mathbb{Z}G)$ is conjugate to an element of $G$ – here $G$ is considered in a natural way as subgroup of $V(\mathbb{Z}G)$ and its elements are then called trivial units of $\mathbb{Z}G$. The conjecture ZC–1 is certainly one of the major open questions for integral group rings and if it is valid for a specific group $G$ it provides of course a positive answer to the prime graph question for $V(\mathbb{Z}G)$.

In Section 2 we reduce the study of the prime graph question to the study of nonabelian composition factors and their automorphism groups. This gives many calculations made in the last years with respect to simple groups (see e.g. [2, 3, 9]) a new value. These calculations are not only examples. In particular with respect to sporadic simple groups they might be the only way to a general result. We note that not only simple groups have to be checked but also their automorphism groups have to be examined which has been done up to now only very rarely.

In Section 3 we consider groups whose order is only divisible by three primes. The prime graph question has a positive answer for all simple groups $G$ of this type and for all their groups of automorphisms sandwiched between $G$ and $\text{Aut}G$ except possibly the cases of $M_{10}$ and $\text{PGL}(2,9) \cong A6.2_2$. For these two groups we are not yet able to show that the normalized unit group of its integral group ring has no elements of order 6.

2. The reduction to almost simple groups

An almost simple group is a subgroup of the automorphism group of a finite non-abelian simple group which contains $\text{Inn}G \cong G$. The object of this section is the following result which reduces the prime graph question to the study of almost simple groups.

Theorem 2.1. Let $G$ be a finite group. Assume that for each almost simple group $X$ which occurs as image of $G$ the prime graph question has a positive answer. Then the prime graph question has a positive answer for $G$.

The first step for the proof is the following.
Proposition 2.2. [14] Proposition 4.3] Let

\[ 1 \longrightarrow A \longrightarrow E \xrightarrow{\phi} G \longrightarrow 1 \]

be a short exact sequence of groups. Assume that \( A \) is a \( p \)-group. Then

\[ \Pi(V(ZG)) = \Pi(G) \implies \Pi(V(ZE)) = \Pi(E). \]

So it remains to study what happens under minimal perfect extensions.

Proposition 2.3. Let

\[ 1 \longrightarrow N \longrightarrow E \xrightarrow{\phi} G \longrightarrow 1 \]

be a short exact sequence of groups. Assume that \( N \) is a perfect minimal normal subgroup of \( G \). Let \( q \in \pi(E \setminus \pi(N)) \) and \( p \in \pi(E) \) with \( q \neq p \). Then \( V(ZE) \) has elements of order \( p \cdot q \) if and only if \( E \) has elements of this order.

Proof. Note first that if \( G \) has elements of order \( p \cdot q \) then \( E \) has elements of order \( p \cdot q \). Assume that \( V(ZE) \) has a unit \( u \) of order \( p \cdot q \) but \( E \) does not have an element of this order. Then \( G \) has no elements of order \( p \cdot q \) and it follows from the assumption that \( \hat{\phi}(u^q) = 1 \) or \( \hat{\phi}(u^q) = 1 \), where \( \hat{\phi} \) denotes the homomorphism from \( V(ZE) \) to \( V(ZG) \) induced from \( \phi \).

Let \( v \) be a non-trivial torsion unit of \( ZE \). By [17, Theorem 2.7] the partial augmentations of \( v \) for a conjugacy class of elements whose order is divisible by a prime not dividing the order of \( V \) are zero. By S.D. Berman [1] and G. Higman [12,18] the 1-coefficient of non-trivial torsion units with augmentation 1 has to be zero. It follows that torsion elements of prime order in the kernel of \( \hat{\phi} \) divide the order of \( N \). Because of the assumption that \( q \) does not divide \( |N| \) we must have \( \hat{\phi}(u^q) = 1 \).

For a group \( X \) and a prime \( r \) denote by \( X(r) \) the set of all elements of \( X \) which order is a positive power of \( r \). Write \( u \) as

\[ u = \sum_{g \in E(p)} a_g g + \sum_{g \in E(q)} b_g g + \sum_{g \in R} c_g g, \]

where \( R = E \setminus (E(p) \cup E(q)) \). Clearly

\[ \hat{\phi}(u) = \sum_{g \in E(p)} a_g \phi(g) + \sum_{g \in E(q)} b_g \phi(g) + \sum_{g \in R} c_g \phi(g), \]

By the previous \( \hat{\phi}(u) \) has order \( q \). Thus by [17] \( u \) has on conjugacy classes of elements in \( R \) partial augmentation zero. If \( x \in E(p) \) and \( \phi(x) \neq 1 \) then the partial augmentation of \( \phi(x) \) is zero. Thus the sum of partial augmentations of \( y \in E \) with \( \phi(y) = x \) is zero. Note that \( y \) is either an element of \( E(p) \) or of \( R \). Finally, if \( x \in E(q) \), then \( \phi(x) \neq 1 \) because \( q \) does not divide \( |N| \). Thus the 1-coefficient of \( \phi(u) \) is the sum of partial augmentations of elements of \( E(p) \) and \( R \). Again by Berman and Higman the 1-coefficient of \( \phi(u) \) is zero. Summarizing, we get that

\[ \sum_{g \in E(p)} a_g + \sum_{g \in R} c_g = 0. \]

(Note that the arguments even show that \( \sum_{g \in E(p)} a_g = \sum_{g \in R} c_g = 0. \) However this fact will not be required for the proof of the proposition.)

Because normalized units have augmentation 1 it follows that

\[ (*) \sum_{g \in E(q)} b_g = 1. \]

Now \( u^q \) has order \( p \) and clearly

\[ u^q = \sum_{g \in E(p)} a_g g^q + \sum_{g \in E(q)} b_g g^q + \sum_{g \in R} c_g g^q \mod [ZE, ZE] + qZE. \]
Again by Berman and Higman and by [17] applied now to \( u^q \) it follows that the sum over the coefficients of \( u^q \) of all elements of order a positive power of \( q \) is also zero. Hence we see that

\[
\sum_{g \in E(q)} b_g \equiv 0 \pmod{q}.
\]

This contradiction to (*) completes the proof. \( \square \)

**Remarks.** a) Instead of [17, Theorem 2.7] we could have used the stronger result [9, Theorem 2.3] which says that partial augmentations of a torsion unit of \( \mathbb{Z}A \) and for \( N \), normal subgroup element of order \( p \) the smallest Suzuki group Sz(8).

b) A typical example for an application of Proposition 2.3 is the automorphism group of the arithmetic of torsion units in integral group rings. However in this article we restrict ourselves on (PQ). Modifications of the arguments for Proposition 2.3 will lead to stronger results on the possible orders of torsion units in integral group rings. However in this article we restrict ourselves on (PQ).

**Proof of Theorem 2.1.** Clearly Theorem 2.1 holds when \( G \) itself is almost simple.

Thus we may apply induction on the length of a chief series. If \( G \) has a minimal normal subgroup \( N \) which is abelian we see by Proposition 2.2 that the Theorem is valid provided it holds for \( G/N \).

Suppose that all minimal normal subgroups of \( G \) are perfect. Let \( p, q \in \pi(G) \) be different primes and suppose that \( V(\mathbb{Z}G) \) has an element of order \( p \cdot q \). By Proposition 2.3 it follows that \( G \) has an element of order \( p \cdot q \) provided \( p \in \pi(G) \setminus \pi(G/N) \) or \( q \in \pi(G) \setminus \pi(G/N) \) for at least one minimal normal subgroup \( N \). So assume that \( p \) and \( q \) divide the order of each minimal normal subgroup of \( G \).

If \( G \) has at least two different minimal normal subgroups \( N_1, N_2 \) then obviously \( G \) has elements of order \( p \cdot q \) because \( N_1 \times N_2 \) is a subgroup of \( G \). The same argument applies if \( G \) has a minimal normal perfect subgroup which is not simple because such a subgroup is a direct product of at least two copies of isomorphic simple groups.

Thus the only case which remains is that \( G \) has a unique minimal normal subgroup which is a non-abelian simple group \( S \) and \( p \) and \( q \) divide the order of \( S \). But this case holds by assumption. \( \square \)

### 3. Groups of orders divisible by three primes only

The goal of this section is the following result.

**Theorem 3.1.** Let \( G \) be a finite group of order \( p^aq^br^c \), where \( p, q, \) and \( r \) are primes. Then \( \Pi(V(\mathbb{Z}G)) = \Pi(G) \) except possibly the case that \( M_{10} \) or \( PGL(2,9) = A_6.2_4 \) are involved in \( G \).

**Proof.** By [11] the simple groups of order divisible by three primes only are

\[
PSL(2,5) \cong A_5, PSL(2,7), PSL(2,8), PSL(2,9) \cong A_6
\]

\[
PSL(2,17), PSL(3,3), PSP(3,4) \cong U(4,2), U(3,3).
\]

For some of these groups and their automorphism groups the first Zassenhaus conjecture ZC–1 holds. This has been proved for \( A_5 \) and \( S_5 \) by Luthar and Passi [15] and by Luthar and Trama [16] respectively. Hertweck verified the prime graph question for \( PSL(2,p), p \) a prime [9] and for \( A_6 \) [10].

For the remaining cases the Luthar-Passi-Hertweck algorithm (also called the HeLP-method) yields the following.

**Case 1.** Let \( G = PSL(2,8) \). Then \( |G| = 504 = 2^3 \cdot 3^2 \cdot 7 \) and \( exp(G) = 126 = 2 \cdot 3^2 \cdot 7 \). The group \( G \) has elements of orders 2, 3, 7, and 9. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 2 and 3 are rationally conjugate to an element of \( G \).
For torsion units of order 7 we obtain the system of inequalities
\[\begin{align*}
\mu_1(u, \chi_7, \nu) &= \frac{1}{6}(5\nu - 2\nu + 2\nu + 9) \geq 0; \\
\mu_2(u, \chi_7, \nu) &= \frac{1}{6}(-2\nu + 5\nu + 5\nu + 9) \geq 0; \\
\mu_3(u, \chi_7, \nu) &= \frac{1}{6}(-2\nu + 2\nu + 5\nu + 9) \geq 0; \\
\mu_1(u, \chi_2, 2) &= \frac{1}{6}(5\nu - 2\nu + 2\nu + 2) \geq 0; \\
\mu_2(u, \chi_2, 2) &= \frac{1}{6}(-2\nu + 5\nu - 2\nu + 2) \geq 0; \\
\mu_3(u, \chi_2, 2) &= \frac{1}{6}(-2\nu - 5\nu + 5\nu + 2) \geq 0; \\
\mu_1(u, \chi_5, 2) &= \frac{1}{6}(3\nu + 3\nu - 4\nu + 4) \geq 0; \\
\mu_2(u, \chi_5, 2) &= \frac{1}{6}(-4\nu + 3\nu + 3\nu + 4) \geq 0; \\
\mu_3(u, \chi_5, 2) &= \frac{1}{6}(3\nu - 4\nu + 3\nu + 4) \geq 0;
\end{align*}\]

which has only three solutions:
\[\{\nu, \nu, \nu\} \in \{(1, 0, 0), (0, 0, 1), (0, 1, 0)\}.
\]

For torsion units of order 9, first in inequalities
\[\begin{align*}
\mu_3(u, \chi_2, \nu) &= \frac{1}{6}(6\nu - 3\nu - 3\nu + 3) \geq 0; \\
\mu_3(u, \chi_2, \nu) &= \frac{1}{6}(12\nu + 6\nu + 6\nu + 3) \geq 0;
\end{align*}\]
put \(t_1 = 2\nu - \nu - \nu - \nu\), then \(t_1 = -1\).

Next, in inequalities
\[\begin{align*}
\mu_1(u, \chi_2, 2) &= \frac{1}{6}(+6\nu - 3\nu - 3\nu + 3) \geq 0; \\
\mu_1(u, \chi_3, \nu) &= \frac{1}{6}(-6\nu + 3\nu + 3\nu + 6) \geq 0;
\end{align*}\]
put \(t_2 = +2\nu - \nu - \nu - \nu\), then \(t_2 \in \{-1, 2\}\).

Finally, in inequalities
\[\begin{align*}
\mu_2(u, \chi_3, \nu) &= \frac{1}{6}(+3\nu - 6\nu + 3\nu + 6) \geq 0; \\
\mu_2(u, \chi_2, 2) &= \frac{1}{6}(-3\nu + 6\nu - 3\nu + 3) \geq 0; \\
\end{align*}\]
put \(t_3 = +\nu - 2\nu + \nu\), then \(t_3 \in \{-2, 1\}\).

After adding the condition \(\nu + \nu + \nu = 1\), solving the system of linear equations for each possible combination of \(t_1, t_2\) and \(t_3\) and using the additional inequality \(\mu_4(u, \chi_2, 2) = \frac{1}{6}(-3\nu + 3\nu + 6\nu + 3) \geq 0;\)
we obtain only three possible solutions for partial augmentations of torsion units of order 9:
\[\{\nu, \nu, \nu\} \in \{(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0)\}.
\]

Thus, torsion units of orders 2, 3, 7 and 9 are rationally conjugate to group elements. It remains to show that no other orders of torsion units are possible in \(\mathbb{Z}G\), and it suffices to show this for orders 6, 14 and 21.

For elements of order 6, the system of inequalities
\[\begin{align*}
\mu_0(u, \chi_2, \nu) &= \frac{1}{6}(-2\nu + 4\nu + 2) \geq 0; \\
\mu_1(u, \chi_2, \nu) &= \frac{1}{6}(-2\nu + 2\nu + 10) \geq 0; \\
\mu_3(u, \chi_2, \nu) &= \frac{1}{6}(2\nu + 4\nu + 4) \geq 0; \\
\nu_0(u, \chi_3, \nu) &= \frac{1}{6}(-2\nu + 2\nu + 8) \geq 0;
\end{align*}\]
has no solutions.
For elements of order 14, we use \((2,7)\)-constant characters approach (cf. \cite{3}). Taking the ordinary character \(\chi_2\) with \(\chi_2(1) = 7, \chi_2(C_2) = -1, \chi_2(C_7) = 0\), we obtain the system
\[
\begin{align*}
\mu_0(u, \chi_2, 0) &= \frac{1}{14}(-6\nu_2 + 6) \geq 0; \\
\mu_7(u, \chi_2, 0) &= \frac{1}{14}(6\nu_2 + 8) \geq 0; \\
\mu_1(u, \chi_2, 0) &= \frac{1}{14}(\nu_2 + 8) \geq 0;
\end{align*}
\]
which has no solutions.

Similarly, for elements of order 21 we use the same character \(\chi_2\) to construct the system
\[
\begin{align*}
\mu_0(u, \chi_2, 0) &= \frac{1}{21}(-24\nu_3 + 3) \geq 0; \\
\mu_7(u, \chi_2, 0) &= \frac{1}{21}(12\nu_3 + 9) \geq 0; \\
\mu_1(u, \chi_2, 0) &= \frac{1}{21}(-2\nu_3 + 9) \geq 0;
\end{align*}
\]
which has no solutions.

Therefore, the 1st Zassenhaus conjecture holds for \(PSL(2,8)\).

**Case 2.** Let \(G = PSL(3,3)\). Then \(|G| = 5616 = 2^4 \cdot 3^3 \cdot 137\) and \(exp(G) = 312 = 2^3 \cdot 3 \cdot 13\). The group \(G\) has elements of orders 2, 3, 4, 6, 8 and 13. By \cite{8} (Proposition 3.1), it follows immediately that torsion units of orders 2 are rationally conjugate to an element of \(G\).

Using the Luthar–Passi–Hertweck method, we are able to show the rational conjugacy for torsion units of orders 4, 8 and 13, but we are unable to eliminate non-trivial solutions for orders 3 and 6. However, for the prime graph question, we need only to show that \(V(ZG)\) has no elements of orders 26 and 39.

For elements of order 26, we use the \((2,13)\)-constant ordinary character \(\chi_3\) of degree 13, with \(\chi_3(C_2) = -3\) and \(\chi(C_{13}) = 0\). Then the system
\[
\begin{align*}
\mu_0(u, \chi_3, 0) &= \frac{1}{39}(-36\nu_2 + 10) \geq 0; \\
\mu_{13}(u, \chi_3, 0) &= \frac{1}{39}(36\nu_2 + 16) \geq 0; \\
\mu_1(u, \chi_3, 0) &= \frac{1}{39}(-3\nu_2 + 16) \geq 0;
\end{align*}
\]
has no solutions.

For elements of order 39, we use the \((3,13)\)-constant ordinary character \(\chi_8\) of degree 26, with \(\chi(C_3) = -1\) and \(\chi(C_{13}) = 0\). It yields
\[
\begin{align*}
\mu_0(u, \chi_8, 0) &= \frac{1}{39}(-24\nu_3 + 24) \geq 0; \\
\mu_{13}(u, \chi_8, 0) &= \frac{1}{39}(12\nu_3 + 27) \geq 0; \\
\mu_1(u, \chi_8, 0) &= \frac{1}{39}(-1\nu_3 + 27) \geq 0;
\end{align*}
\]
and this system has no solutions either.

**Case 3.** Let \(G = PSP(3,4) \cong U(4,2)\). Then \(|G| = 25920 = 2^6 \cdot 3^4 \cdot 5\) and \(exp(G) = 180 = 2^2 \cdot 3^2 \cdot 5\). The group \(G\) has elements of orders 2, 3, 4, 5, 6, 9 and 12. By \cite{8} (Proposition 3.1), it follows immediately that torsion units of order 5 are rationally conjugate to an element of \(G\). To give the positive answer to the (PQ) we need to show that \(V(ZG)\) has no elements of orders 10 and 15.

For elements of order 10, we use ordinary characters \(\chi_9\) of degree 20 with \(\chi_9(C_2) = 4\) and \(\chi_9(C_5) = 0\), and \(\chi_{17}\) of degree 45 with \(\chi_{17}(C_2) = -3\) and \(\chi_{17}(C_5) = 0\). Then the system
\[
\begin{align*}
\mu_0(u, \chi_9, 0) &= \frac{1}{10}(16\nu_2 + 24) \geq 0; \\
\mu_5(u, \chi_9, 0) &= \frac{1}{10}(-16\nu_2 + 16) \geq 0; \\
\mu_1(u, \chi_{17}, 0) &= \frac{1}{10}(-3\nu_2 + 48) \geq 0;
\end{align*}
\]
has no solutions.
For elements of order 15, first we use the ordinary characters $\chi_{11}$ of degree 30 with $\chi_{11}(C_3) = 3$ and $\chi_{11}(C_5) = 0$. Then from

$$
\mu_0(u, \chi_{11}, 0) = \frac{1}{15}(24\nu_3 + 36) \geq 0; \\
\mu_2(u, \chi_{11}, 0) = \frac{1}{15}(-12\nu_3 + 27) \geq 0;
$$

it follows that $\nu_3 = 1$, so $\nu_5 = 0$. Now using the ordinary character $\chi_{20}$ of degree 81 such that $\chi_{20}(C_3) = 0$ and $\chi_{20}(C_5) = 1$, we obtain that $\mu_0(u, \chi_{20}, 0) = \frac{1}{15}(8\nu_5 + 85) \geq 0$ is not an integer when $\nu_5 = 0$.

**Case 4.** Let $G = U(3,3)$. Then $|G| = 6048 = 2^5 \cdot 3^3 \cdot 7$ and $exp(G) = 168 = 2^3 \cdot 3 \cdot 7$. The group $G$ has elements of orders 2, 3, 4, 6, 7, 8 and 12. By [8] (Proposition 3.1), it follows immediately that torsion units of order 2, 3, 4, 6, 7 and 8 are rationally conjugate to an element of $G$. To give the positive answer to the (PQ) we need to show that $V(ZG)$ has no elements of orders 14 and 21.

For elements of order 14, consider the ordinary character $\chi_4$ of degree 7 with $\chi(C_2) = 3$, $\chi(C_7) = 0$. Then the system of constraints

$$
\mu_0(u, \chi_4, 0) = \frac{1}{14}(18\nu_2 + 10) \geq 0; \\
\mu_1(u, \chi_4, 0) = \frac{1}{14}(3\nu_2 + 4) \geq 0; \\
\mu_7(u, \chi_4, 0) = \frac{1}{14}(-18\nu_2 + 4) \geq 0;
$$

has no solutions.

For elements of order 21, consider the ordinary character $\chi_{10}$ of degree 27 with $\chi_{10}(C_3) = 0$, $\chi_{10}(C_7) = -1$. Then the system

$$
\mu_0(u, \chi_{10}, 0) = \frac{1}{21}(-12\nu_7 + 21) \geq 0; \\
\mu_1(u, \chi_{10}, 0) = \frac{1}{21}(-1\nu_7 + 28) \geq 0; \\
\mu_7(u, \chi_{10}, 0) = \frac{1}{21}(6\nu_7 + 21) \geq 0;
$$

has no solutions.

Now we need to look at automorphism groups, as specified in the next table (we use the same notations for groups as in the GAP Character Table Library [5]). Only the case of $S_5$ is covered by Luthar and Trama in [10]. For the remaining groups, we will investigate the prime graph question (PQ) below. Note that for some groups the results are stronger.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\text{Aut}(G)$</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PSL(2,5) \cong A_5$</td>
<td>$S_5$</td>
<td>i.e. to check: $S_6, A6.2_4, M10, A6.2_2$</td>
</tr>
<tr>
<td>$PSL(2,7)$</td>
<td>$PSL(3,2) : C2$</td>
<td></td>
</tr>
<tr>
<td>$PSL(2,8)$</td>
<td>$PSL(2,8).3$</td>
<td></td>
</tr>
<tr>
<td>$PSL(2,9) \cong A_6$</td>
<td>$(A6.C_2) : C2$</td>
<td></td>
</tr>
<tr>
<td>$PSL(2,17)$</td>
<td>$PSL(2,17).2$</td>
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</tr>
<tr>
<td>$PSL(3,3)$</td>
<td>$PSL(3,3).2$</td>
<td></td>
</tr>
<tr>
<td>$PSP(3,4) \cong U(4,2)$</td>
<td>$U(4,2).2$</td>
<td></td>
</tr>
<tr>
<td>$U(3,3)$</td>
<td>$U(3,3).2$</td>
<td></td>
</tr>
</tbody>
</table>

- Let $G = \text{Aut}(PSL(2,7)) = PSL(3,2) : C2$. Then $|G| = 336 = 2^4 \cdot 3 \cdot 7$ and $exp(G) = 168 = 2^3 \cdot 3 \cdot 7$. The group $G$ has elements of orders 2, 3, 4, 6, 7 and 8. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 3 and 7 are rationally conjugate to an element of $G$. For this automorphism group we are able to show that the first Zassenhaus conjecture holds, and to do this it remains to prove rational conjugacy for torsion units of orders 2, 4, 6 and 8, and then show that $V(ZG)$ has no elements of orders 12, 14 and 21.

For units of order 2, the system of inequalities

$$
\mu_0(u, \chi_2, *) = \frac{1}{2}(\nu_{2a} - \nu_{2b} + 1) \geq 0; \\
\mu_1(u, \chi_2, *) = \frac{1}{2}(-\nu_{2a} + \nu_{2b} + 1) \geq 0;
$$

has only two trivial solutions $\{\nu_{2a}, \nu_{2b}\} \in \{(1,0), (0,1)\}$. 


For units of order 4, the system

\[
\begin{align*}
\mu_0(u, \chi_2, \ast) &= \frac{1}{2}(2\nu_{2a} + 2\nu_{4a} - 2\nu_{2b} + 2) \geq 0; \\
\mu_2(u, \chi_2, \ast) &= \frac{1}{2}(-2\nu_{2a} - 2\nu_{4a} + 2\nu_{2b} + 2) \geq 0; \\
\mu_0(u, \chi_3, \ast) &= \frac{1}{2}(-4\nu_{2a} + 4\nu_{4a} + 4) \geq 0; \\
\mu_2(u, \chi_3, \ast) &= \frac{1}{2}(4\nu_{2a} - 4\nu_{4a} + 4) \geq 0;
\end{align*}
\]

has only three solutions \((\nu_{2a}, \nu_{4a}, \nu_{2b}) \in \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}\). After the Cohn-Livingstone test (Theorem 4.1, [3]) it remains only one solution \((\nu_{2a}, \nu_{4a}, \nu_{2b}) = (0, 1, 0)\).

For units of order 6 we need to consider two cases dependently on whether \(\chi(u^3) = \chi(2a)\) or \(\chi(u^3) = \chi(2b)\).

When \(\chi(u^3) = \chi(2a)\), we obtain the system

\[
\begin{align*}
\mu_0(u, \chi_3, \ast) &= \frac{1}{2}(-4\nu_{2a} + 4) \geq 0; \\
\mu_2(u, \chi_3, \ast) &= \frac{1}{2}(2\nu_{2a} + 4) \geq 0; \\
\mu_0(u, \chi_8, \ast) &= \frac{1}{2}(-2\nu_{3a} + 4\nu_{2b} - 2\nu_{6a} + 6) \geq 0; \\
\mu_1(u, \chi_8, \ast) &= \frac{1}{2}(-\nu_{3a} + 2\nu_{2b} - \nu_{9a} + 9) \geq 0; \\
\mu_3(u, \chi_8, \ast) &= \frac{1}{2}(2\nu_{3a} - 4\nu_{2b} + 2\nu_{6a} + 6) \geq 0; \\
\mu_0(u, \chi_3, 7) &= \frac{1}{2}(-2\nu_{2a} - 2\nu_{2b} + 4\nu_{6a} + 2) \geq 0; \\
\mu_0(u, \chi_4, 7) &= \frac{1}{2}(-2\nu_{2a} + 2\nu_{2b} - 4\nu_{6a} + 2) \geq 0; \\
\mu_0(u, \chi_5, 7) &= \frac{1}{2}(2\nu_{2a} - 2\nu_{3a} + 2\nu_{2b} + 2\nu_{6a} + 4) \geq 0;
\end{align*}
\]

which has no solutions.

When \(\chi(u^3) = \chi(2b)\), we get the system

\[
\begin{align*}
\mu_0(u, \chi_3, \ast) &= \frac{1}{2}(-4\nu_{2a} + 6) \geq 0; \\
\mu_3(u, \chi_3, \ast) &= \frac{1}{2}(4\nu_{2a} + 6) \geq 0; \\
\mu_0(u, \chi_8, \ast) &= \frac{1}{2}(-2\nu_{3a} + 4\nu_{2b} - 2\nu_{6a} + 8) \geq 0; \\
\mu_1(u, \chi_8, \ast) &= \frac{1}{2}(-\nu_{3a} + 2\nu_{2b} - \nu_{9a} + 7) \geq 0; \\
\mu_3(u, \chi_8, \ast) &= \frac{1}{2}(2\nu_{3a} - 4\nu_{2b} + 2\nu_{6a} + 4) \geq 0;
\end{align*}
\]

which has only one trivial solution \((\nu_{2a}, \nu_{3a}, \nu_{2b}, \nu_{6a}) = (0, 0, 0, 1)\).

For elements of order 8, we have to consider two cases: \(\chi(u^4) = \chi(2a)\) and \(\chi(u^4) = \chi(2b)\).

When \(\chi(u^4) = \chi(2a)\), we get the system of inequalities

\[
\begin{align*}
\mu_0(u, \chi_2, \ast) &= \frac{1}{8}(4\nu_{2a} + 4\nu_{4a} - 4\nu_{2b} - 4\nu_{8a} - 4\nu_{8b} + 4) \geq 0; \\
\mu_4(u, \chi_2, \ast) &= \frac{1}{8}(-4\nu_{2a} - 4\nu_{4a} + 4\nu_{2b} + 4\nu_{8a} + 4\nu_{8b} + 4) \geq 0; \\
\mu_0(u, \chi_3, \ast) &= \frac{1}{8}(-8\nu_{2a} + 8\nu_{4a} + 8) \geq 0; \\
\mu_4(u, \chi_3, \ast) &= \frac{1}{8}(8\nu_{2a} - 8\nu_{4a} + 8) \geq 0; \\
\mu_1(u, \chi_4, \ast) &= \frac{1}{8}(4\nu_{8a} - 4\nu_{8b} + 4) \geq 0; \\
\mu_3(u, \chi_4, \ast) &= \frac{1}{8}(-4\nu_{8a} + 4\nu_{8b} + 4) \geq 0; \\
\mu_0(u, \chi_6, \ast) &= \frac{1}{8}(-4\nu_{2a} - 4\nu_{4a} + 4\nu_{2b} - 4\nu_{8a} - 4\nu_{8b} + 4) \geq 0; \\
\mu_4(u, \chi_6, \ast) &= \frac{1}{8}(4\nu_{2a} + 4\nu_{4a} - 4\nu_{2b} + 4\nu_{8a} + 4\nu_{8b} + 4) \geq 0; \\
\mu_0(u, \chi_7, \ast) &= \frac{1}{8}(-4\nu_{2a} - 4\nu_{4a} - 4\nu_{2b} + 4\nu_{8a} + 4\nu_{8b} + 4) \geq 0;
\end{align*}
\]

which has only two trivial solutions \((\nu_{2a}, \nu_{4a}, \nu_{2b}, \nu_{8a}, \nu_{8b}) \in \{(0, 0, 0, 0, 1), (0, 0, 0, 1, 0)\}\).

When \(\chi(u^4) = \chi(2b)\), we get that \(\mu_1(u, \chi_2, 0) = 1/4\) is not an integer, so this case is not possible.

Thus, now we proved rational conjugacy for elements of all orders that appear in \(G\), and it remains only to show that \(V(ZG)\) has no elements of orders 12, 14 and 21.
For elements of order 12, we need to consider two cases. If \( \chi(u^6) = \chi(2a) \), then \( \mu_1(u, \chi_2, 0) = -1/6 \) is not an integer. If \( \chi(u^6) = \chi(2b) \), then \( \mu_3(u, \chi_2, 0) = 1/2 \) is not an integer. Therefore, no elements of order 12 can appear in \( V(ZG) \).

For the case of units of order 14, we use the ordinary character \( \chi_7 \) of degree 7 such that \( \chi(C_2) = -1, \chi(C_7) = 0 \) to construct the system

\[
\begin{align*}
\mu_0(u, \chi_7, 0) &= \frac{1}{14}(-6\nu_2 + 6) \geq 0; \\
\mu_1(u, \chi_7, 0) &= \frac{1}{14}(\nu_2 + 8) \geq 0; \\
\mu_7(u, \chi_7, 0) &= \frac{1}{14}(6\nu_2 + 8) \geq 0; \\
\end{align*}
\]

which has no solutions.

For the case of units of order 21, we use the ordinary character \( \chi_6 \) of degree 7 such that \( \chi(C_3) = 1, \chi(C_7) = 0 \) to construct the system

\[
\begin{align*}
\mu_0(u, \chi_6, 0) &= \frac{1}{21}(12\nu_3 + 9) \geq 0; \\
\mu_1(u, \chi_6, 0) &= \frac{1}{21}(\nu_3 + 6) \geq 0; \\
\mu_7(u, \chi_6, 0) &= \frac{1}{21}(-6\nu_3 + 6) \geq 0; \\
\end{align*}
\]

which has no solutions.

Thus, the Zassenhaus conjecture ZC–1 holds for \( G \).

- Let \( G = Aut(PSL(2,8)) = PSL(2,8) \). Then \( |G| = 1512 = 2^4 \cdot 3^3 \cdot 7 \) and \( exp(G) = 126 = 2 \cdot 3^2 \cdot 7 \). The group \( G \) has elements of orders 2, 3, 6, 7 and 9. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 2 and 7 are rationally conjugate to an element of \( G \). For this automorphism group we are able to prove the rational conjugacy for all orders that appear in \( G \) except the order 6, and then confirm the (IP-C) conjecture, showing that \( V(ZG) \) has no elements of orders 14, 18 and 21.

For elements of order 3, the system

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{4}(2\nu_{3a} - \nu_{3b} - \nu_{3c} + 1) \geq 0; \\
\mu_1(u, \chi_2, *) &= \frac{1}{4}(-\nu_{3a} + 2\nu_{3b} - \nu_{3c} + 1) \geq 0; \\
\mu_2(u, \chi_2, *) &= \frac{1}{4}(-\nu_{3a} - \nu_{3b} + 2\nu_{3c} + 1) \geq 0; \\
\mu_0(u, \chi_4, *) &= \frac{1}{4}(-4\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} + 7) \geq 0; \\
\mu_1(u, \chi_4, *) &= \frac{1}{4}(2\nu_{3a} - \nu_{3b} - \nu_{3c} + 7) \geq 0; \\
\mu_2(u, \chi_4, *) &= \frac{1}{4}(-4\nu_{3a} - \nu_{3b} - \nu_{3c} + 7) \geq 0; \\
\mu_0(u, \chi_5, *) &= \frac{1}{4}(2\nu_{3a} + 2\nu_{3b} - \nu_{3c} + 7) \geq 0; \\
\mu_1(u, \chi_5, *) &= \frac{1}{4}(2\nu_{3a} - \nu_{3b} + 2\nu_{3c} + 7) \geq 0; \\
\mu_2(u, \chi_5, *) &= \frac{1}{4}(2\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} + 7) \geq 0; \\
\mu_0(u, \chi_7, *) &= \frac{1}{4}(6\nu_{3a} + 21) \geq 0; \\
\mu_1(u, \chi_7, *) &= \frac{1}{4}(-3\nu_{3a} + 21) \geq 0; \\
\mu_0(u, \chi_8, *) &= \frac{1}{4}(2\nu_{3a} + 4\nu_{3b} + 4\nu_{3c} + 8) \geq 0; \\
\mu_1(u, \chi_8, *) &= \frac{1}{4}(\nu_{3a} - 2\nu_{3b} - 2\nu_{3c} + 8) \geq 0; \\
\mu_1(u, \chi_9, *) &= \frac{1}{4}(\nu_{3a} + 4\nu_{3b} - 2\nu_{3c} + 8) \geq 0; \\
\mu_2(u, \chi_9, *) &= \frac{1}{4}(\nu_{3a} - 2\nu_{3b} + 4\nu_{3c} + 8) \geq 0; \\
\mu_0(u, \chi_4, 2) &= \frac{1}{4}(-6\nu_{3a} + 6) \geq 0; \\
\mu_1(u, \chi_4, 2) &= \frac{1}{4}(3\nu_{3a} + 6) \geq 0; \\
\mu_0(u, \chi_5, 2) &= \frac{1}{4}(6\nu_{3a} + 12) \geq 0; \\
\mu_1(u, \chi_5, 2) &= \frac{1}{4}(-3\nu_{3a} + 12) \geq 0;
\end{align*}
\]

has only three trivial solutions \((\nu_{3a}, \nu_{3b}, \nu_{3c}) \in \{(1,0,0), (0,0,1), (0,1,0)\}\).

For units of order 6, we have to consider three cases for \( \chi(u^2) \in \{ \chi(3a), \chi(3b), \chi(3c) \} \).
For $\chi(u^2) = \chi(3a)$, we have the system
\[
\mu_0(u, \chi_4, *) = \frac{1}{6}(-2\nu_{2a} - 4\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} - 2\nu_{6a} - 2\nu_{6b} + 2) \geq 0;
\]
\[
\mu_1(u, \chi_4, *) = \frac{1}{6}(-2\nu_{2a} - 2\nu_{3a} + \nu_{3b} + \nu_{3c} - \nu_{6a} - \nu_{6b} + 10) \geq 0;
\]
\[
\mu_3(u, \chi_4, *) = \frac{1}{6}(2\nu_{2a} + 4\nu_{3a} - 2\nu_{3b} - 2\nu_{3c} + 2\nu_{6a} + 2\nu_{6b} + 4) \geq 0;
\]
\[
\mu_0(u, \chi_5, *) = \frac{1}{6}(-2\nu_{2a} - 4\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 2) \geq 0;
\]
\[
\mu_1(u, \chi_5, *) = \frac{1}{6}(-\nu_{2a} - 2\nu_{3a} + \nu_{3b} - 2\nu_{3c} - \nu_{6a} + 2\nu_{6b} + 10) \geq 0;
\]
\[
\mu_3(u, \chi_5, *) = \frac{1}{6}(2\nu_{2a} + 4\nu_{3a} + \nu_{3b} + 2\nu_{3c} - \nu_{6a} - \nu_{6b} + 4) \geq 0;
\]
\[
\mu_4(u, \chi_5, *) = \frac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{6a} - 2\nu_{6b} + 8) \geq 0;
\]
\[
\mu_0(u, \chi_7, *) = \frac{1}{6}(-6\nu_{2a} + 6\nu_{3a} + 24) \geq 0;
\]
which has 4 solutions
\[
(\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{6a}, \nu_{6b}) \in \{(2, -1, 1, 2, -2), (2, -1, 0, 1, 0, -1), (2, -1, 1, 0, -1, 0), (2, -1, 2, -1, -2, 1)\}.
\]

When $\chi(u^2) = \chi(3b)$, the system
\[
\mu_0(u, \chi_4, *) = \frac{1}{6}(-2\nu_{2a} - 4\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} - 2\nu_{6a} - 2\nu_{6b} + 8) \geq 0;
\]
\[
\mu_2(u, \chi_4, *) = \frac{1}{6}(2\nu_{2a} + 2\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 5) \geq 0;
\]
\[
\mu_0(u, \chi_5, *) = \frac{1}{6}(-2\nu_{2a} - 4\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 5) \geq 0;
\]
\[
\mu_1(u, \chi_5, *) = \frac{1}{6}(\nu_{2a} + 2\nu_{3a} + 2\nu_{3b} - \nu_{3c} - 2\nu_{6a} + \nu_{6b} + 5) \geq 0;
\]
\[
\mu_2(u, \chi_5, *) = \frac{1}{6}(2\nu_{2a} + 4\nu_{3a} + \nu_{3b} + 2\nu_{3c} - \nu_{6a} - \nu_{6b} + 7) \geq 0;
\]
\[
\mu_4(u, \chi_5, *) = \frac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{6a} - \nu_{6b} + 7) \geq 0;
\]
\[
\mu_0(u, \chi_7, *) = \frac{1}{6}(-6\nu_{2a} + 6\nu_{3a} + 18) \geq 0;
\]
has 9 solutions:
\[
(\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{6a}, \nu_{6b}) \in \{(-2, 2, -1, 2, 1, -1), (-2, 2, 0, 1, 0, 0), (-2, 2, 1, 0, -1, 1), (0, 0, 0, 0, 0, 1), (0, 0, 0, 3, 0, -2), (0, 1, 1, -1, 1, 0), (0, 0, 1, 2, -1, -1), (0, 0, 2, 1, -2, 0)\}
\]

Finally, when $\chi(u^2) = \chi(3c)$, we obtain the system
\[
\mu_0(u, \chi_4, *) = \frac{1}{6}(-2\nu_{2a} - 4\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} - 2\nu_{6a} - 2\nu_{6b} + 8) \geq 0;
\]
\[
\mu_2(u, \chi_4, *) = \frac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 5) \geq 0;
\]
\[
\mu_0(u, \chi_5, *) = \frac{1}{6}(-2\nu_{2a} - 4\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 5) \geq 0;
\]
\[
\mu_1(u, \chi_5, *) = \frac{1}{6}(\nu_{2a} - 2\nu_{3a} + \nu_{3b} - 2\nu_{3c} - \nu_{6a} + 2\nu_{6b} + 7) \geq 0;
\]
\[
\mu_3(u, \chi_5, *) = \frac{1}{6}(2\nu_{2a} + 4\nu_{3a} + \nu_{3b} + 2\nu_{3c} - \nu_{6a} - \nu_{6b} + 7) \geq 0;
\]
\[
\mu_4(u, \chi_5, *) = \frac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{6a} - 2\nu_{6b} + 5) \geq 0;
\]
\[
\mu_0(u, \chi_7, *) = \frac{1}{6}(-6\nu_{2a} + 6\nu_{3a} + 18) \geq 0;
\]
which has 9 solutions
\[
(\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{6a}, \nu_{6b}) \in \{(-2, 2, 0, 1, 1, -1), (-2, 2, 1, 0, 0, 0), (-2, 2, 2, -1, -1, 1), (0, 0, -1, 1, 2, -1), (0, 0, 0, 0, 1, 0), (0, 0, 1, -1, 0, 1), (0, 0, 3, 0, -2, 0)\}
\]

For units of order 9, again we have to consider three cases for $\chi(u^2) \in \{\chi(3a), \chi(3b), \chi(3c)\}$. 
For \( \chi(u^2) = \chi(3a) \), we have the system
\[
\mu_0(u, \chi_2, *) = \frac{1}{9}(6\nu_{3a} + 6\nu_{9a} - 3\nu_{2b} - 3\nu_{3c} - 3\nu_{9b} - 3\nu_{9c} + 3) \geq 0;
\mu_3(u, \chi_2, *) = \frac{1}{9}(-3\nu_{3a} - 3\nu_{9a} + 6\nu_{2b} - 3\nu_{3c} + 6\nu_{9b} - 3\nu_{9c} + 3) \geq 0;
\mu_0(u, \chi_4, *) = \frac{1}{9}(-12\nu_{3a} + 6\nu_{9a} + 6\nu_{2b} + 6\nu_{3c} + 6\nu_{9b} + 6\nu_{9c} + 3) \geq 0;
\mu_3(u, \chi_4, *) = \frac{1}{9}(6\nu_{3a} - 3\nu_{9a} - 3\nu_{2b} - 3\nu_{3c} - 3\nu_{9b} - 3\nu_{9c} + 3) \geq 0;
\mu_0(u, \chi_5, *) = \frac{1}{9}(-12\nu_{3a} + 6\nu_{9a} - 3\nu_{2b} - 3\nu_{3c} - 3\nu_{9b} - 3\nu_{9c} + 3) \geq 0;
\mu_3(u, \chi_5, *) = \frac{1}{9}(6\nu_{3a} - 3\nu_{9a} - 3\nu_{2b} - 3\nu_{3c} + 6\nu_{9b} - 3\nu_{9c} + 3) \geq 0;
\mu_0(u, \chi_7, *) = \frac{1}{9}(18\nu_{3a} + 27) \geq 0;
\mu_3(u, \chi_7, *) = \frac{1}{9}(-9\nu_{3a} + 27) \geq 0;
\mu_0(u, \chi_8, *) = \frac{1}{9}(-6\nu_{3a} - 6\nu_{9a} + 12\nu_{2b} + 12\nu_{3c} - 6\nu_{9b} - 6\nu_{9c} + 6) \geq 0;
\mu_3(u, \chi_8, *) = \frac{1}{9}(3\nu_{3a} + 3\nu_{9a} - 6\nu_{2b} - 6\nu_{3c} + 3\nu_{9b} + 3\nu_{9c} + 6) \geq 0;
\mu_0(u, \chi_9, *) = \frac{1}{9}(-6\nu_{3a} - 6\nu_{9a} - 6\nu_{2b} - 6\nu_{3c} + 3\nu_{9b} + 3\nu_{9c} + 6) \geq 0;
\mu_3(u, \chi_9, *) = \frac{1}{9}(3\nu_{3a} + 3\nu_{9a} + 12\nu_{2b} - 6\nu_{3c} + 6\nu_{9b} + 3\nu_{9c} + 6) \geq 0;
\mu_0(u, \chi_7, 2) = \frac{1}{9}(-18\nu_{3a}) \geq 0;
\mu_3(u, \chi_7, 2) = \frac{1}{9}(9\nu_{3a}) \geq 0;
\mu_0(u, \chi_5, 2) = \frac{1}{9}(18\nu_{3a} - 18\nu_{9a} + 18) \geq 0;
\mu_3(u, \chi_5, 2) = \frac{1}{9}(9\nu_{3a} + 9\nu_{9a} + 18) \geq 0;
\]
which has 5 solutions, only three of which remain after the Cohn-Livingstone test [6]:
\[
(\nu_{3a}, \nu_{9a}, \nu_{2b}, \nu_{3c}, \nu_{9b}, \nu_{9c}) \in \{(0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0), (0, 1, 0, 0, 0, 0)\}.
\]

If \( \chi(u^3) = \chi(3b) \), then \( \mu_1(u, \chi_2, 0) = 1/3 \) is not an integer, so this case is impossible. Similarly, when \( \chi(u^3) = \chi(3c) \), then \( \mu_2(u, \chi_2, 0) = 1/3 \), so this case is not possible as well.

Now, after we have covered orders that appear in the group \( G \), we need to show that \( V(ZG) \) has no elements of orders 14, 18 and 21.

For elements of order 14, we use the ordinary character \( \chi_4 \) of degree 7 with \( \chi(C_2) = -1, \chi(C_7) = 0 \). Then the system
\[
\mu_0(u, \chi_4, 0) = \frac{1}{17}(-6\nu_2 + 6) \geq 0;
\mu_1(u, \chi_4, 0) = \frac{1}{17}(-1\nu_2 + 8) \geq 0;
\mu_7(u, \chi_4, 0) = \frac{1}{17}(6\nu_2 + 8) \geq 0;
\]
has no solutions.

For elements of order 18, we need to consider \( 22 \times 3 \times 3 = 198 \) cases determined by combinations of \( \chi(u^6), \chi(u^2) \) and \( \chi(u^3) \). Using the development version of the GAP package LAGUNA [4] we verified that each case leads to a contradiction since there is certain \( \mu_i(u, \chi_j, 0) \) which is not an integer. We omit full details here (but may provide a table summarising the results, if need be).

For elements of order 21, we use the ordinary character \( \chi_{11} \) of degree 27 with \( \chi(C_3) = 0, \chi(C_7) = -1 \). Then the system
\[
\mu_0(u, \chi_{11}, 0) = \frac{1}{27}(-12\nu_7 + 21) \geq 0;
\mu_1(u, \chi_{11}, 0) = \frac{1}{27}(-\nu_7 + 28) \geq 0;
\mu_7(u, \chi_{11}, 0) = \frac{1}{27}(6\nu_7 + 21) \geq 0;
\]
has no solutions.
• Let $G = \text{Aut}(\text{PSL}(2,9)) = (A_6 C_2) : C_2$. For this group we need to consider four possible cases. Here, as well as in the rest of the proof, we give only the outline, specifying which orders of torsion units we considered. The detailed proof will be included in the full version of the paper.

**Case 1.** Let $G = S_6$. Then $|G| = 720 = 2^4 \cdot 3^2 \cdot 5$ and $\text{exp}(G) = 60 = 2^2 \cdot 3 \cdot 5$. By [8] (Proposition 3.1), it follows immediately that torsion units of order 5 are rationally conjugate to an element of $G$. For this group we are able to confirm the (IP-C) conjecture, showing that there are no elements of orders 10, 12 and 15 in $V(ZG)$.

**Case 2.** Let $G = A_6 2$.$ Then $|G| = 720 = 2^4 \cdot 3^2 \cdot 5$ and $\text{exp}(G) = 120 = 2^3 \cdot 3 \cdot 5$. By [8] (Proposition 3.1), it follows immediately that torsion units of order 3 are rationally conjugate to an element of $G$. For the remaining orders, we are able to prove the rational conjugacy for all orders that appear in $G(2,3,4,5,8$ and $10)$, and also to show that there are no elements of orders 15 and 20 in $V(ZG)$. However, we are not yet able to eliminate the tuple $(\nu_{2a}, \nu_{3a}, \nu_{2b}) = (-2, 3, 0)$ for torsion units of order 6, so (PQ) still remains open for this group.

**Case 3.** Let $G = M_{10}$. Then $|G| = 720 = 2^4 \cdot 3^2 \cdot 5$ and $\text{exp}(G) = 120 = 2^3 \cdot 3 \cdot 5$. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 2, 3 and 5 are rationally conjugate to an element of $G$. We can also prove rational conjugacy for the order 4, but not for the order 8. For orders that do not appear in $G$, we are able to show that there are no elements of orders 10 and 15 in $V(ZG)$, however, we are not yet able to eliminate the tuple $(\nu_{2a}, \nu_{3a}) = (-2, 3)$ for torsion units of order 6, so (PQ) still remains open for this group.

**Case 4.** Let $G = A_6 2^2$. In this case using the HeLP - method we are able to give positive answer to (PQ) eliminating the order 15.

For the remaining four automorphism groups the HeLP - method answers (PQ) positively, eliminating the following orders of torsion units of $V(ZG)$: orders 34 and 51 for $\text{Aut}(\text{PSL}(2,17)) = \text{PGL}(2,17)$; orders 26 and 39 for $\text{Aut}(\text{PSL}(3,3)) = \text{PSL}(3,3)$; order 15 for $\text{Aut}(\text{PSP}(3,4)) = U(4,2) 2$; and finally, orders 14 and 21 for $\text{Aut}(U(3,3)) = U(3,3) 2$. This completes the proof.

**References**


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