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On certain exotic 4-manifolds of Akhmedov and Park

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ON CERTAIN EXOTIC 4-MANIFOLDS OF AKHMEDOV AND PARK

ABSTRACT. In an article from 2008, A. Akhmedov and B. D. Park constructed irreducible symplectic 4-manifolds homeomorphic but not diffeomorphic to the manifolds $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ and $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$. These manifolds are constructed by using generalized fibre sums. In this note we describe an explicit splitting of the second (co-)homology of these manifolds adapted to their construction as fibre sums. We also calculate the canonical classes of the symplectic structures. This gives a new proof for a formula derived by A. Akhmedov, R. İ. Baykur and B. D. Park.

CONTENTS

1. Introduction	1
2. Generalized fibre sums	1
3. Construction of the exotic $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$	2
4. Construction of the exotic $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$	6
References	9

1. INTRODUCTION

In their article [4], A. Akhmedov and B. D. Park constructed simply-connected irreducible symplectic 4-manifolds U and V homeomorphic to $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ and $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$. In particular, being irreducible under connected sum, U and V are not diffeomorphic to these manifolds and hence determine exotic smooth structures. The construction of these manifolds uses the generalized fibre sum, that we will recall in Section 2. It turns out that, even though the building blocks in this construction have non-trivial fundamental group, the manifolds U and V obtained as multiple fibre sums are simply-connected.

The second (co-)homology of a generalized fibre sum of two 4-manifolds admits a canonical decomposition, see [8]. In this note we want to describe this splitting for the manifolds U and V . This will also allow us to determine the canonical classes of the symplectic structures (a formula for the canonical class has also been derived in a different way in [3]). It is also quite easy to see with these methods that the manifolds U and V have vanishing first integral homology. We will, however, not repeat the difficult part in [4] and show that U and V are simply-connected. To understand why the manifolds U and V have vanishing first integral homology without going through the calculation of their fundamental groups was one of the starting points for the following discussion. In general we will see that the calculations on the level of homology are fairly simple.

2. GENERALIZED FIBRE SUMS

In the following, we use for a topological space Y the abbreviations $H_*(Y)$ and $H^*(Y)$ to denote the homology and cohomology groups of Y with \mathbb{Z} -coefficients. The homology class of an embedded,

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oriented surface and the surface itself are often denoted by the same symbol. Poincaré duality is often suppressed, so that a class and its Poincaré dual are denoted by the same symbol.

Let M and N be smooth, closed, oriented 4-manifolds and Σ_M and Σ_N closed, oriented, embedded surfaces of the same genus g in M and N . We assume that both surfaces represent homology classes of self-intersection zero. Then their normal bundles are trivial. We choose trivializations of their normal bundles corresponding to framings of the tubular neighbourhoods as $\nu\Sigma_M = \Sigma_M \times D^2$ and $\nu\Sigma_N = \Sigma_N \times D^2$. Let $\phi : \Sigma_M \rightarrow \Sigma_N$ be an orientation preserving diffeomorphism. Then the generalized fibre sum of M and N along Σ_M and Σ_N is defined as

$$X = M' \cup_{\Phi} N'$$

where M' and N' denote the manifolds with boundary $\Sigma_M \times S^1$ and $\Sigma_N \times S^1$ obtained by deleting the interior of the tubular neighbourhoods of the surfaces and Φ is an orientation reversing diffeomorphism $\Phi : \partial M' \rightarrow \partial N'$ that preserves the S^1 fibration, reverses orientation on the circles and covers the diffeomorphism ϕ between the surfaces. For example in the fixed framing we can consider the diffeomorphism

$$(1) \quad \begin{aligned} \Phi : \Sigma_M \times S^1 &\rightarrow \Sigma_N \times S^1, \\ (x, z) &\mapsto (\phi(x), \bar{z}), \end{aligned}$$

where the bar denotes complex conjugation. In general, the smooth structure of the 4-manifold X can depend on the choice of the covering diffeomorphism Φ . We choose one such covering and denote the resulting manifold by

$$X = M \#_{\phi} N.$$

We can specify the diffeomorphism $\phi : \Sigma_M \rightarrow \Sigma_N$ equivalently by choosing embeddings

$$\begin{aligned} i_M : \Sigma_g &\rightarrow M \\ i_N : \Sigma_g &\rightarrow N \end{aligned}$$

that realize the surfaces Σ_M and Σ_N as images of a fixed closed surface Σ_g of genus g . Then we choose as ϕ the diffeomorphism $i_N \circ (i_M^{-1}|_{\Sigma_M})$.

If M and N are symplectic manifolds and Σ_M, Σ_N symplectically embedded, then the generalized fibre sum X admits a symplectic structure. See references [6] and [9] for the original construction.

3. CONSTRUCTION OF THE EXOTIC $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$

Let K be a knot in S^3 . Denote a tubular neighbourhood of K by $\nu K \cong S^1 \times D^2$. Let m be a fibre of the circle bundle $\partial\nu K \rightarrow K$ and use an oriented Seifert surface for K to define a section $l : K \rightarrow \partial\nu K$. The circles m and l are called the *meridian* and the *longitude* of K . Let M_K be the closed 3-manifold obtained by 0-surgery on K , defined as

$$M_K = (S^3 \setminus \text{int } \nu K) \cup_f (S^1 \times D^2).$$

where the gluing diffeomorphism f maps in homology ∂D^2 onto the longitude l . One can show that M_K has the same integral homology as $S^2 \times S^1$. The meridian m , which bounds the fibre in the normal bundle to K in S^3 , becomes non-zero in the homology of M_K and defines a generator in $H_1(M_K)$. The longitude l is null-homotopic in M_K since it bounds one of the D^2 -fibres glued in. This copy of D^2 determines together with the Seifert surface of K a closed, oriented surface in M_K which intersects m once and generates $H_2(M_K)$.

Let K be the left-handed trefoil knot in S^3 . Then according to equation (1) in [4], the fundamental group of M_K is given by

$$\pi_1(M_K) = \langle a, b \mid aba = bab, a^{-1}b^{-2}a^{-1}b^4 = 1 \rangle.$$

Hence in homology we have again $H_1(M_K) = \mathbb{Z}$ generated by $a = b$ which are equal, under appropriate choice of orientation, to the class of the generator m .

The knot K is a genus one fibred knot, hence M_K is a T^2 -bundle over S^1 . It has a section $S^1 \rightarrow M_K$ whose image is equal to b . Consider the 4-manifold $S^1 \times M_K$. It is a T^2 -bundle over T^2 with section $S = x \times b$, where x denotes the S^1 -factor. In particular, the inclusion induced map $H_1(S) \rightarrow H_1(S^1 \times M_K)$ is an isomorphism. Let F denote a fibre of this bundle. Then according to [4] we can write $F = \gamma_1 \times \gamma_2$ where the curves γ_1 and γ_2 are homotopic to $a^{-1}b$ and $b^{-1}aba^{-1}$. In homology, both γ_1 and γ_2 vanish, hence the inclusion induced homomorphism $H_1(F) \rightarrow H_1(S^1 \times M_K)$ is the zero map. Both S and F are embedded tori of self-intersection zero. We have $H_2(S^1 \times M_K) = \mathbb{Z}^2$, generated by S and F . The intersection form is given in this basis by

$$Q_{S^1 \times M_K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By a construction of W. P. Thurston [10], the manifold $S^1 \times M_K$ admits a symplectic structure such that both S and F are symplectic submanifolds.

Lemma 1. *The canonical class of the symplectic manifold $S^1 \times M_K$ is given by $K_{S^1 \times M_K} = 0$.*

Proof. This follows from the adjunction formula

$$2g - 2 = \Sigma_g^2 + K\Sigma_g,$$

since both generators S and F are symplectic. □

Consider two copies of $S^1 \times M_K$ and embeddings

$$\begin{aligned} i_1 &: T^2 \rightarrow S^1 \times M_K \\ i_2 &: T^2 \rightarrow S^1 \times M_K \end{aligned}$$

whose images are S and F and which map the standard generators of $\pi_1(T^2)$ to $\{x, b\}$ and $\{\gamma_1, \gamma_2\}$, respectively.

Definition 2. Let Y_K denote the symplectic fibre sum $(S^1 \times M_K) \#_{\psi} (S^1 \times M_K)$, where the diffeomorphism $\psi : S \rightarrow F$ is given by $i_2 \circ i_1^{-1}$.

In other words [5], Y_K is obtained by knot surgery with the left-handed trefoil knot on $S^1 \times M_K$ along the fibre F . It is known that knot surgery does not change the integral homology groups and the intersection form on H_2 and that rim tori do not exist in the knot surgered manifold (this also follows with the methods in [8]). Hence we have:

Lemma 3. $H_1(Y_K) = \mathbb{Z}^2$ and $H_2(Y_K) = \mathbb{Z}^2$.

The calculation of the first homology of Y_K of course also follows from the calculation of its fundamental group in [4].

The generators of $H_2(Y_K)$ can be described as follows: We can consider push-offs of the surface S in the first copy and F in the second copy of $S^1 \times M_K$ into the boundary of their tubular neighbourhoods. If we choose the gluing diffeomorphism as in equation (1), then both push-offs get identified to a torus T_{Y_K} of self-intersection zero inside Y_K . This is one generator of $H_2(Y_K)$. The second generator is a surface Σ of genus 2 and self-intersection zero in Y_K obtained by sewing together a

punctured fibre from the first copy and a punctured section from the second copy of $S^1 \times M_K$. By the Gompf construction [6] we can assume that Σ is symplectic. The intersection form on the generators T_{Y_K} and Σ is given by

$$Q_{Y_K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 4. *The canonical class of the symplectic manifold Y_K is given by $K_{Y_K} = 2T_{Y_K}$.*

Proof. This follows from the adjunction formula since both generators T_{Y_K} and Σ are symplectic. \square

We can also describe the inclusion induced map $H_1(\Sigma) \rightarrow H_1(Y_K)$. Consider the following part of the Mayer-Vietoris sequence for Y_K :

$$\dots \rightarrow H_1(T^2 \times S^1) \rightarrow H_1(S^1 \times M_K \setminus \nu S) \oplus H_1(S^1 \times M_K \setminus \nu F) \rightarrow H_1(Y_K) \rightarrow 0.$$

In $S^1 \times M_K \setminus \nu S$ we have the punctured fibre and in $S^1 \times M_K \setminus \nu F$ the punctured section which sew together to define the surface Σ . Since $S \cdot F = 1$, both the section and the fibre represent indivisible elements in homology. This implies that the meridians to these surfaces are zero in the homology of the complements of the tubular neighbourhoods and we have isomorphisms

$$H_1(S^1 \times M_K \setminus \nu S) \cong H_1(S^1 \times M_K \setminus \nu F) \cong H_1(S^1 \times M_K).$$

The Mayer-Vietoris sequence reduces to

$$H_1(T^2) \xrightarrow{i_1 \oplus i_2} H_1(S^1 \times M_K) \oplus H_1(S^1 \times M_K) \rightarrow H_1(Y_K) \rightarrow 0.$$

Hence $H_1(Y_K)$ is isomorphic to the cokernel of $i_1 \oplus i_2$. The map i_1 on homology is an isomorphism, whereas the map i_2 is the zero map. It follows that the inclusion maps the generators of the punctured section to the generators of $H_1(Y_K)$ and the generators of the punctured fibre to zero. In the notation of [4], the group $H_1(Y_K)$ has generators y, d and the inclusion maps the standard generators of $H_1(\Sigma)$ to $\{y, d, 0, 0\}$ in that particular order.

The manifold Y_K is the first building block for U . The second building block is the manifold $Q = (S^1 \times M_K) \# 2\overline{\mathbb{C}P^2}$. In Q there is a symplectic surface Σ' of genus 2 and self-intersection zero, obtained by symplectically resolving the intersection point of a torus fibre F and a torus section S in $S^1 \times M_K$ and then blowing up at two points.

Let h, z denote the generators of $H_1(Q) = H_1(S^1 \times M_K) = \mathbb{Z}^2$ corresponding to the generators b, x we had previously. Then the inclusion maps the standard generators of $H_1(\Sigma')$ to $\{z, h, 0, 0\}$ in that particular order. Choose embeddings of a reference surface of genus 2

$$\begin{aligned} i_{Y_K} : \Sigma_2 &\rightarrow Y_K \\ i_Q : \Sigma_2 &\rightarrow Q \end{aligned}$$

whose images are Σ and Σ' and which map the standard generators of $H_1(\Sigma_2)$ to $\{y, d, 0, 0\}$ and $\{0, 0, z, h\}$, respectively.

Definition 5. Let U denote the fibre sum $Y_K \#_{\phi} Q$, where the diffeomorphism $\phi : \Sigma \rightarrow \Sigma'$ is given by $i_Q \circ i_{Y_K}^{-1}$.

Proposition 6. *Rim tori do not exist in the fibre sum U . The 4-manifold U is a homology $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$.*

Proof. We use the results from [8]. According to [8, Corollary 45], $H_1(U)$ is isomorphic to the cokernel of the map $i_{Y_K} \oplus i_Q : H_1(\Sigma_2) \rightarrow H_1(Y_K) \oplus H_1(Q)$. Since this map is an isomorphism, $H_1(U) = 0$. Similarly, according to [8, Theorem 51], the subgroup of rim tori in the second homology of U is isomorphic to the cokernel of the map $i_{Y_K}^* + i_Q^* : H^1(Y_K) \oplus H^1(Q) \rightarrow H^1(\Sigma_2)$. Since this

map is also an isomorphism, rim tori do not occur in the 4-manifold U . Finally, the formulae in [8, Corollary 40] show that $b_2^+(U) = 1$ and $b_2^-(U) = 3$. \square

The 4-manifold U is symplectic, since the surfaces Σ and Σ' are symplectically embedded. In [4] the gluing diffeomorphism ϕ is specified on the level of fundamental groups and it is shown that U is simply-connected and irreducible. Hence the manifold U is an exotic $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$.

We now describe the splitting of $H_2(U)$ adapted to the fibre sum. We decompose the second homology of the manifold Q as

$$H_2(Q) = \mathbb{Z}\Sigma' \oplus \mathbb{Z}B_Q \oplus P(Q),$$

where B_Q is a surface in Q with $\Sigma' \cdot B_Q = 1$ and $P(Q)$ denotes the orthogonal complement of $\mathbb{Z}\Sigma' \oplus \mathbb{Z}B_Q$ with respect to the intersection form. The direct sum decomposition holds, because the intersection form is unimodular on the subgroup $\mathbb{Z}\Sigma' \oplus \mathbb{Z}B_Q$, see [7, Lemma 1.2.12]. Similarly, we have a decomposition

$$H_2(Y_K) = \mathbb{Z}\Sigma \oplus \mathbb{Z}T_{Y_K}.$$

In this case the subgroup $P(Y_K)$ is zero. The push-offs of the surfaces Σ and Σ' determine a surface Σ_U in U of genus 2 and self-intersection 0. The punctured surfaces B_Q and T_{Y_K} sew together to define a surface B_U of genus equal to the genus of B_Q plus one. The surface B_U has self-intersection B_Q^2 since the torus T_{Y_K} has self-intersection 0. Since rim tori and the dual vanishing (or *split*) classes do not exist in U , [8, Theorem 59] shows that

$$H_2(U) = \mathbb{Z}\Sigma_U \oplus \mathbb{Z}B_U \oplus P(Q).$$

The subgroup $P(Q)$ is orthogonal to the first two summands. The restriction of the intersection form to $\mathbb{Z}\Sigma_U \oplus \mathbb{Z}B_U$ is of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & B_Q^2 \end{pmatrix}$$

and the intersection form on $P(Q)$ is the one induced from Q . Note that there is an isomorphism $H_2(U) \cong H_2(Q)$ preserving the intersection form obtained by mapping Σ' to Σ_U , B_Q to B_U and the identity on $P(Q)$.

We now determine the canonical class of U , which depends on the choice of the surface B_Q .

Proposition 7. *Let E_1, E_2 denote the exceptional spheres in Q . Then the canonical classes of the symplectic 4-manifolds Q and U are given by*

$$K_Q = E_1 + E_2$$

and

$$K_U = (2 + K_Q B_Q - 2B_Q^2)\Sigma_U + 2B_U + (K_Q - 2B_Q - (K_Q B_Q - 2B_Q^2)\Sigma').$$

In the formula for K_U the term $K_Q - 2B_Q - (K_Q B_Q - 2B_Q^2)\Sigma'$ is an element of $P(Q)$.

Proof. The formula for K_Q follows from Lemma 1 and the adjunction formula for the exceptional spheres. According to [8, Theorem 89], the canonical class of U is given by

$$K_U = \overline{K_Q} + b_U B_U + (\eta_U + \eta'_U)\Sigma_U,$$

where

$$\begin{aligned}\overline{K_Q} &= K_Q - (2g - 2)B_Q - (K_Q B_Q - (2g - 2)B_Q^2)\Sigma' \in P(Q) \\ b_U &= 2g - 2 \\ \eta_U &= K_{Y_K} T_{Y_K} + 1 - (2g - 2)T_{Y_K}^2 \\ \eta'_U &= K_Q B_Q + 1 - (2g - 2)B_Q^2.\end{aligned}$$

In our case, $g = 2$ and K_{Y_K} is given by Lemma 4. \square

For instance, we can choose as B_Q the section S or the fibre F in $S^1 \times M_K$. In both cases $B_Q^2 = 0$ and $K_Q B_Q = 0$, hence

$$K_U = 2\Sigma_U + 2B_U + (K_Q - 2B_Q).$$

With the formula for the intersection form of U it follows that $K_U^2 = 6$, as expected from the formula $K_U^2 = 2e(U) + 3\sigma(U)$.

4. CONSTRUCTION OF THE EXOTIC $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$

The first building block for the exotic 4-manifold V is $R = T^4 \# 2\overline{\mathbb{C}P^2}$. Fix a factorization $T^4 = T^2 \times T^2$ and choose a symplectic structure on T^4 such that both tori are symplectically embedded. Symplectically resolving the intersection point of the two tori and blowing up twice we obtain a symplectic surface Σ'' of genus 2 and self-intersection zero in R .

Let $\alpha_i, i = 1, \dots, 4$, denote the generator of $H_1(R) = H_1(T^4)$ corresponding to i -th circle factor. Then the inclusion maps the standard generators of $H_1(\Sigma'')$ to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in that particular order. In particular, the inclusion induced map $H_1(\Sigma'') \rightarrow H_1(R)$ is an isomorphism.

To describe the second building block of the manifold V , we consider two copies of the manifold Y_K constructed above. Recall that in Y_K there is a symplectic surface Σ of genus 2 and self-intersection zero. If y, d denote the generators of $H_1(Y_K)$ then the inclusion maps the standard generators of $H_1(\Sigma)$ to $\{y, d, 0, 0\}$. Let t, s denote generators of the second copy of Y_K corresponding to y, d . Choose embeddings of a reference surface of genus 2

$$\begin{aligned}i_{Y_{K1}} &: \Sigma_2 \rightarrow Y_K \\ i_{Y_{K2}} &: \Sigma_2 \rightarrow Y_K\end{aligned}$$

whose images are the surfaces Σ in the first and second copy of Y_K and which map the standard generators of $H_1(\Sigma_2)$ to $\{y, d, 0, 0\}$ and $\{0, 0, t, s\}$, respectively.

Definition 8. Let X_K denote the symplectic fibre sum $Y_K \#_{\psi} Y_K$, where the diffeomorphism $\psi : \Sigma \rightarrow \Sigma$ is given by $i_{Y_{K2}} \circ i_{Y_{K1}}^{-1}$.

Lemma 9. *Rim tori do not exist in the fibre sum X_K . We have $H_1(X_K) = 0$ and $H_2(X_K) = \mathbb{Z}^2$.*

Proof. By construction, the map $i_{Y_{K1}} \oplus i_{Y_{K2}} : H_1(\Sigma_2) \rightarrow H_1(Y_K) \oplus H_1(Y_K)$ is an isomorphism. Hence $H_1(X_K)$, which is isomorphic to the cokernel of this map, vanishes. Similarly, $i_{Y_{K1}}^* + i_{Y_{K2}}^* : H^1(Y_K) \oplus H^1(Y_K) \rightarrow H^1(\Sigma_2)$ is an isomorphism. Therefore, rim tori do not exist in the fibre sum X_K . Finally, the claim that $b_2(X_K) = 2$ follows from [8, Corollary 40]. \square

This lemma has also been proved in [1]. We can describe the splitting of the second homology of X_K adapted to the fibre sum as follows. The second homology of the first copy of Y_K splits as

$$H_2(Y_K) = \mathbb{Z}\Sigma \oplus \mathbb{Z}T_{Y_K}.$$

and similarly for the second copy. The push-offs of the surfaces Σ in the first and second copy determine a symplectic surface Σ_{X_K} of genus 2 and self-intersection zero in X_K . The punctured tori T_{Y_K} in the first and second copy of Y_K sew together to determine a surface B_{X_K} of genus 2 and self-intersection zero in X_K . By the Gompf construction we can assume that B_{X_K} is symplectic. Since rim tori and vanishing classes do not exist in Y_K , we have

$$H_2(X_K) = \mathbb{Z}\Sigma_{X_K} \oplus \mathbb{Z}B_{X_K}.$$

The intersection form in this basis is given by

$$Q_{X_K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

hence X_K is a homology $S^2 \times S^2$.

Lemma 10. *The canonical class of the symplectic manifold X_K is given by $K_{X_K} = 2\Sigma_{X_K} + 2B_{X_K}$.*

Proof. This follows from the adjunction formula since both surfaces Σ_{X_K} and B_{X_K} are symplectic. \square

We choose embeddings of a reference surface of genus 2

$$\begin{aligned} i_R : \Sigma_2 &\rightarrow R \\ i_{X_K} : \Sigma_2 &\rightarrow X_K \end{aligned}$$

whose images are the surfaces Σ'' and Σ_{X_K} and which map the standard generators of $H_1(\Sigma_2)$ to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\{0, 0, 0, 0\}$, respectively.

Definition 11. Let V denote the symplectic fibre sum $R \#_{\phi} X_K$, where the diffeomorphism $\phi : \Sigma'' \rightarrow \Sigma_{X_K}$ is given by $i_{X_K} \circ i_R^{-1}$.

Proposition 12. *Rim tori do not exist in the fibre sum V . The 4-manifold V is a homology $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$.*

Proof. By construction, the map $i_R \oplus i_{X_K} : H_1(\Sigma_2) \rightarrow H_1(R) \oplus H_1(X_K) = H_1(R)$ is an isomorphism. Hence $H_1(V)$, which is isomorphic to the cokernel of this map, vanishes. Similarly, $i_R^* + i_{X_K}^* : H^1(R) \oplus H^1(X_K) = H^1(R) \rightarrow H^1(\Sigma_2)$ is an isomorphism. Therefore, rim tori do not exist in the fibre sum V . Finally, the claim that $b_2^+(V) = 3$ and $b_2^-(V) = 5$ follows again from [8, Corollary 40]. \square

In [4] it is shown that V is simply-connected and irreducible. Hence the manifold V is an exotic $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$.

We describe the splitting of $H_2(V)$ adapted to the fibre sum. We first decompose the second homology of the manifold R as

$$H_2(R) = \mathbb{Z}\Sigma'' \oplus \mathbb{Z}B_R \oplus P(R),$$

where B_R is a surface in R with $\Sigma'' \cdot B_R = 1$ and $P(R)$ denotes the orthogonal complement of $\mathbb{Z}\Sigma'' \oplus \mathbb{Z}B_R$ with respect to the intersection form. We also have a decomposition

$$H_2(X_K) = \mathbb{Z}\Sigma_{X_K} \oplus \mathbb{Z}B_{X_K},$$

where both Σ_{X_K} and B_{X_K} are surfaces of genus 2 and self-intersection zero. The push-offs of the surfaces Σ'' and Σ_{X_K} determine a surface Σ_V in V of genus 2 and self-intersection 0. The punctured surfaces B_R and B_{X_K} sew together to define a surface B_V of genus equal to the genus of B_R plus

two. The surface B_V has self-intersection B_R^2 . Since rim tori and the dual vanishing classes do not exist in V , [8, Theorem 59] shows that

$$H_2(V) = \mathbb{Z}\Sigma_V \oplus \mathbb{Z}B_V \oplus P(R).$$

The subgroup $P(R)$ is orthogonal to the first two summands. The restriction of the intersection form to $\mathbb{Z}\Sigma_V \oplus \mathbb{Z}B_V$ is of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & B_R^2 \end{pmatrix}$$

and the intersection form on $P(R)$ is the one induced from R . There is again an isomorphism $H_2(V) \cong H_2(R)$ preserving the intersection form.

We determine the canonical class of V , depending on the choice of the surface B_R .

Proposition 13. *Let E_1, E_2 denote the exceptional spheres in R . Then the canonical classes of the symplectic 4-manifolds R and V are given by*

$$K_R = E_1 + E_2$$

and

$$K_V = (4 + K_R B_R - 2B_R^2)\Sigma_V + 2B_V + (K_R - 2B_R - (K_R B_R - 2B_R^2)\Sigma'').$$

In the formula for K_V the term $K_R - 2B_R - (K_R B_R - 2B_R^2)\Sigma''$ is an element of $P(R)$.

Proof. The first claim follows because $K_{T^4} = 0$. According to [8, Theorem 89], the canonical class of V is given by

$$K_V = \overline{K_R} + b_V B_V + (\eta_V + \eta'_V)\Sigma_V,$$

where

$$\begin{aligned} \overline{K_R} &= K_R - (2g - 2)B_R - (K_R B_R - (2g - 2)B_R^2)\Sigma'' \in P(R) \\ b_V &= 2g - 2 \\ \eta_V &= K_R B_R + 1 - (2g - 2)B_R^2 \\ \eta'_V &= K_{X_K} B_{X_K} + 1 - (2g - 2)B_{X_K}^2. \end{aligned}$$

In our case, $g = 2$ and K_{X_K} is given by Lemma 10. □

For example, we can take as B_R one of the torus factors in $T^4 = T^2 \times T^2$. Then $B_R^2 = 0$ and $K_R B_R = 0$, hence

$$K_V = 4\Sigma_V + 2B_V + (K_R - 2B_R).$$

With the formula for the intersection form we have $K_V^2 = 14$, as expected.

Remark 14. In [2], A. Akhmedov constructed irreducible symplectic 4-manifolds Y and X homeomorphic to $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$ and $3\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$ using generalized fibre sums. The building blocks of X and Y are the manifolds X_K and Y_K and $Z = T^2 \times S^2 \# 4\overline{\mathbb{C}P^2}$. The manifold Z admits a Lefschetz fibration with fibres of genus 2. Let Σ'_2 denote a regular fibre and a_1, b_1 the generators of $H_1(Z)$ in the notation of [2]. Then the inclusion induced homomorphism maps the standard generators of $H_1(\Sigma'_2)$ to $\{a_1, b_1, -a_1, -b_1\}$ in that particular order.

The manifold Y is obtained as a generalized fibre sum of Y_K and Z . Using similar arguments as before one can show that rim tori do not exist in the fibre sum Y and calculate the canonical class. The manifold X , homeomorphic to $3\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$ and obtained as a fibre sum of X_K and Z , however, does contain rim tori. The subgroup of rim tori in the second homology of X is given by the cokernel of the inclusion induced homomorphism $H^1(Z) \rightarrow H^1(\Sigma_2)$ and hence is isomorphic to \mathbb{Z}^2 . There

also exists a dual subgroup of vanishing classes. If $R(X)$ and $S'(X)$ denote the groups of rim tori and vanishing classes, then in a similar way as before

$$H_2(X) = \mathbb{Z}\Sigma_X \oplus \mathbb{Z}B_X \oplus P(Z) \oplus R(X) \oplus S'(X).$$

In this case the canonical class of X contains a rim tori contribution that depends on the choice of covering diffeomorphism Φ used in the construction of the fibre sum. See [8, Theorem 89] for the general formula.

REFERENCES

1. A. Akhmedov, *Construction of symplectic cohomology $S^2 \times S^2$* , Gökova Proc. **11**, 36–48 (2007).
2. A. Akhmedov, *Small exotic 4-manifolds*, Algebr. Geom. Topol. **8**, 1781–1794 (2008).
3. A. Akhmedov, R. İ. Baykur, B. D. Park, *Constructing infinitely many smooth structures on small 4-manifolds*, Journ. Topology **1**, 409–428 (2008).
4. A. Akhmedov, B. D. Park, *Exotic smooth structures on small 4-manifolds*, Invent. Math. **173**, 209–223 (2008).
5. R. Fintushel, R. J. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134**, 363–400 (1998).
6. R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. **142**, 527–595 (1995).
7. R. E. Gompf, A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. Providence, Rhode Island. American Mathematical Society 1999.
8. M. J. D. Hamilton, *Generalized fibre sums of 4-manifolds and the canonical class*, preprint arXiv:0907.2671.
9. J. D. McCarthy, J. G. Wolfson, *Symplectic normal connect sum*, Topology **33**, 729–764 (1994).
10. W. P. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55**, 467–468 (1976).

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